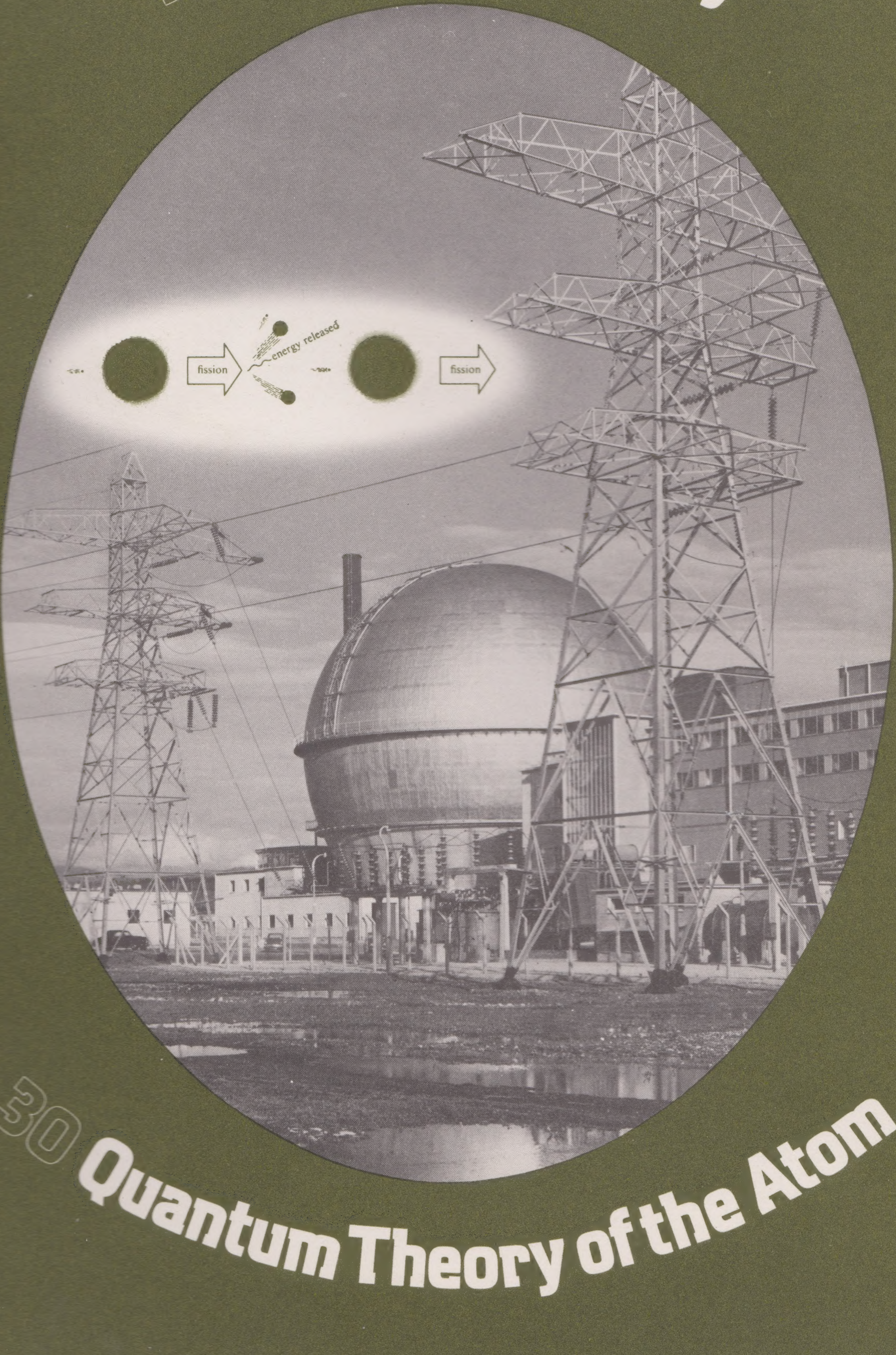
S101 Units 29 and 30

THE OPEN UNIVERSITY

Science : A Foundation Course

29 Quantum Theory



30 Quantum Theory of the Atom

The Open University

Science: A Foundation Course

Unit 29
Quantum Theory

Prepared by the Science Foundation Course Team

The Open University Press

SCIENCE

S101 Course Team List

A note about the authorship of this text

This text is one of a series that constitutes a component part of the Science Foundation Course. The other components are a series of television and radio programmes, audio-vision materials, home experiments, assignments and a summer school.

The Course has been produced by a team that accepts responsibility for it.

The Science Foundation Course Team

Mike Pentz (*Chairman and General Editor*)

Francis Aprahamian (*Course Editor*)
David Broadhurst (*Physics*)
Geoff Brown (*Earth Sciences*)
Neil Chalmers (*Biology*)
Bob Cordell (*Biology*)
Glen Davey (*Course Coordinator*)
Graham Farmelo (*Physics*)
Peter Francis (*Earth Sciences*)
Stuart Freake (*Physics*)
Anna Furth (*Biology*)
Denis Gartside (*BBC*)
Charles Harding (*Chemistry*)
Keith Hodgkinson (*Physics*)
Stephen Hurry (*Biology*)
David Jackson (*BBC*)
David Johnson (*Chemistry*)
Tony Jolly (*BBC*)
Roger Jones (*BBC*)
Patricia McCurry (*BBC*)
Perry Morley (*Editor*)
Jane Nelson (*Chemistry*)
Ian Nuttall (*Editor*)
Irene Ridge (*Biology*)
David Roberts (*Chemistry*)
Shelagh Ross (*Course Coordinator*)
Eileen Scanlon (*Research Fellow, Evaluation*)
Milo Shott (*Physics*)
Jacqueline Stewart (*Editor*)
John Stratford (*BBC*)
Steve Swithenby (*Physics*)
Peggy Varley (*Biology*)
Kiki Warr (*Chemistry*)
Barrie Whatley (*BBC*)
Dave Williams (*Earth Sciences*)
Chris Wilson (*Earth Sciences*)

The following people assisted with particular parts or aspects of the Course:

Tel Bailey (*Tutor*)
Mary Bell (*Biology*)
Frances Berrigan (*IET*)
James Brennan (*Biology, Bridgewater State College, USA*)
Joan Brown (*Earth Sciences*)
Beryl Crooks (*IET*)
Linda Fawke (*Tutor*)
Michael Gagan (*Chemistry*)
Ian Gass (*Earth Sciences*)
Ros Hamilton (*Tutor*)
Robin Harding (*Biology*)
Barbara Hodgson (*IET*)
Jen Horgan (*Tutor*)
Graham Jenkins (*Earth Sciences*)
Barrie Jones (*Physics*)
Paul Lyle (*Tutor*)
John Marshall (*Senior Counsellor*)
Bob Maybury (*Chemistry, UNESCO*)
Bob McConnell (*Earth Sciences, Virginia Polytechnic and the State University, USA*)
Laurie Melton (*Librarian*)
Reg Melton (*IET*)
Yow Lam Oh (*Home Experiments, Caulfield Institute of Technology, Australia*)
Robin Russell (*Chemistry*)
Meg Sheffield (*BBC*)
Jane Sheppard (*Designer*)
Jennie Simmons (*Home Experiment Kit Coordinator*)
Peter Smith (*Earth Sciences*)
Russell Stannard (*Physics*)
Charles Turner (*Earth Sciences*)
Arthur Vialls (*BBC*)
John Walters (*Physics*)
Pete Wood (*Home Experiment Kit Coordinator*)
John Wright (*Earth Sciences*)
Martin Wright (*BBC*)

The Open University Press,
Walton Hall, Milton Keynes.

First published 1979. Reprinted with corrections 1980, 1982.

Designed by the Media Development Group of the Open University.

Typeset by Santype International Limited, Salisbury, Wilts, and printed by Eyre & Spottiswoode Ltd, at Grosvenor Press, Portsmouth.

ISBN 0 355 08066 9

This text forms part of an Open University course. The complete list of Units in the Course is printed at the end of this text.

For general availability of supporting material referred to in this text please write to: Open University Educational Enterprises Limited, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK 11 1BY, Great Britain.

Further information on Open University courses may be obtained from the Admissions Office, The Open University, P.O. Box 48, Walton Hall, Milton Keynes, MK7 6AB.

1.3

Contents

Table A List of terms and concepts in Unit 29

Introduced in a previous Unit	Unit No.	Developed in this Unit	Page No.
amplitude	9	action	16
atomic structure	10–11	Compton scattering	24
conservation of energy	8	components of momentum	9
conservation of momentum	3(TV), 8	conservation of momentum	6
diffraction	9	de Broglie wavelength $\lambda_{dB} = h/p$	13
electron	9, 10–11	Heisenberg's uncertainty principle	39
electromagnetic radiation	9	interactions are described by exchanges of energy and momentum between quanta	26
Fermat's principle	4	Maupertuis' principle (of least action)	17
frequency	9	magnitude of momentum	6
gravitational force, F_g	3	momentum of a photon $p = hf/c$	24
ground state	10–11	philosophical interpretations of quantum theory	40
intensity	9	probability waves	33
mass	3	propagation of any radiation is described by waves	22
molecule	10–11	quantum	25
momentum p	3, 8	quantum theory	25
Newton's laws of motion	3	travelling wave packets	20
nucleus	10–11	travelling wave trains	19
period	1	uncertainty relations:	
photon	9	for position and momentum	35
photoelectric effect	9	for time intervals and energy	38
Planck's constant h	9		
reflection	9		
refraction	9		
speed	3		
velocity	3		
wavelength	9		

Introduction and Study Guide

With this Unit you are entering the last stage of the Science Foundation Course. Compared with the immediately preceding Units—in which you studied the history of the Earth, theories of its origin and the long-term evolution of its structure—it may seem to be an abrupt change of subject. However, when viewed in the wider context of the Course as a whole, there is more than one reason why you are being asked to return to some basic physics now.

Quantum theory, as you will begin to appreciate after you have studied Units 29 and 30, is of fundamental importance to many scientific problems outside physics. Without it, chemists would not be able to study and understand the details of chemical structure of complex substances, or even make sense of the multiplicity of chemical elements. Biologists would be deprived of the use of electron microscopes; and the understanding of the Earth's history would be very much poorer for the lack of information obtained from the dating by radioactive isotopes.

These are just a few examples where progress in other branches of science is derived from quantum physics. At a more practical level, it is not generally appreciated that the birth of pocket calculators—and indeed of the whole fields of computing, automation and long-distance space communications—can be traced back to quantum theory and to its applications in the study of semiconductors.

Important as all these applications are, there is yet another overwhelming reason why you should be introduced to quantum theory now. It is the impact this theory has on the conceptual foundations of science and philosophy. Indeed, it is this aspect of it that the Course Team consider to be the most important one in the context of S101. This Unit will probably shatter some 'common-sense certainties' in your way of thinking about the world. If you should find this process painful, take heart from the fact that some of the best scientific brains struggled with these problems over the whole of the first quarter of this century. Einstein once said about common sense that it is 'that body of prejudices laid down in the mind prior to the age of eighteen'. The one thing that you will need, most of all, this week is the courage to throw away the crutches of your common sense when they come into conflict with experimental evidence and logic.

This Unit has only two components: the Main Text and a television programme, accompanied by TV *Broadcast Notes*. As usual, the Main Text carries the story line, and the TV programme provides visual evidence of some of the experimental results on which the whole theoretical argument rests. The Main Text refers frequently to Units 8 and 9. It would be useful to page through these very quickly before you start working on this Unit, just to refresh your memory. Ideally, you should read through the Main Text before the first viewing of the TV programme. If you could arrange your working schedule so as to fit in a second viewing after you have done some more work on the Main Text, then so much the better.

Note that in Radio 15, we describe the early development of quantum theory and we point out the many deep insights into the structure of matter that the theory has given.

The three Appendices after the Main Text should not be regarded as core material. They will not be assessed in any way, but they should provide an additional challenge to those who may find the Main Text straightforward.

After you have studied this Unit, you will not end up with the full knowledge of quantum theory. And you will not be able to use it as a tool for solving problems. Such aims cannot be achieved without considerable background and training in mathematics. But the Course Team believes that it is possible, with very little use of mathematics, to achieve the Objectives listed at the end of the Main Text and to obtain a worthwhile appreciation of quantum theory along the lines summarized in Section 6. This appreciation will be developed further in Unit 30, where basic principles of quantum theory are applied to the understanding of the structure of atoms and atomic nuclei.

1 How energy and momentum are propagated

You have already met situations where energy is transmitted from one place to another through different media. In particular, you saw in Unit 4 how large-scale disturbances, taking place within the Earth, travel through different layers of materials and are then observed at the surface. It was noted there that the propagation of seismic waves—and, indeed, the propagation of any waves—is governed by Fermat's principle of least time. Taking into account different velocities in different materials, any disturbance that travels from place A to place B in the form of a wave motion, always follows the path that takes the shortest possible *time*. It is not necessarily the path of shortest *length*.

There is another class of phenomena in which propagation and transfer of energy are of importance, of which here are a few examples. Our planet is continuously bombarded by streams of microscopic cosmic particles coming from the Universe. The picture in your television set is produced by a beam of electrons that originates from a hot, metal wire at the thin end of the television tube, is accelerated by an electrostatic field and is finally deflected by variable magnetic fields so that it scans the surface of the screen. In Unit 31, you will hear about accelerators that can produce beams of extremely fast elementary particles for the investigation of their interactions with different targets. In mass spectrographs (Units 10 and 11), different isotopes are identified because beams of ions with different mass-to-charge ratios travel along different paths.

In all these examples, *beams of particles*, carrying energy, travel from one place to another, their propagation being affected by gravitational, electric and magnetic forces. In some cases they simply change their direction and speed, but in others they are completely stopped or they even disappear in interactions with other particles. Is there any pattern in all this behaviour? Can one explain the propagation of particles in terms of some general principle, one that might be analogous to Fermat's principle of least time for the propagation of waves?

Before this question can be answered, we must return for a while to one important concept that has been introduced only very briefly in Units 3 and 8—the concept of momentum.

1.1 Conservation of momentum

Section 1.1 revises some material from Units 3 and 8, and extends the concept of momentum to three dimensions. If you are confident about vectors, in general, and about the conservation of momentum, in particular, you can skim through it very quickly.

In Unit 3, the momentum of an object was defined as the product of its mass and its *velocity*. Notice that according to this definition the momentum has *direction*—the same as the direction of velocity.

This aspect is very important for the full description of motion. Speed only tells you how fast an object is moving, but not in what direction it is travelling. So, just as we have carefully distinguished between velocity and speed in Unit 3, we shall be careful in this Unit to distinguish between, on the one hand, the *momentum*, defined as:

momentum

$$\vec{p} = m \, . \, \vec{v} \qquad (1)$$

(where the arrows indicate that direction is important) and, on the other, the *magnitude of the momentum*, which is simply the product of mass and *speed*:

magnitude of momentum

$$p = m \, . \, v \qquad (1')$$

In the TV programme associated with Unit 3 (TV03), you met for the first time the *law of conservation of momentum*, which was stated as follows:

conservation of momentum

> The total momentum of any group of objects which are not subject to unbalanced external forces is constant. (*Broadcast Notes* for Unit 3)

The TV demonstration was limited to a simple case of two objects: a pair of ice skaters, or a pair of gliders on a frictionless track. The two skaters initially stood still (velocities zero) and then they pushed each other apart. They were seen to be moving in a straight line in opposite directions. Thus their velocities had different signs (one positive, one negative) and the speeds (or, if you like, the magnitudes of their velocities) depended on the masses of the two skaters in such a way that the

total momentum of both skaters was always zero, just as it was before the push.

SAQ 1 The masses of two skaters A and B are $m_A = 80\,\text{kg}$ and $m_B = 50\,\text{kg}$. What is the ratio of the magnitudes of their velocities v_A/v_B after they have pushed each other apart?

(*Hint* Write down the expression for the total momentum of both skaters and compare it with the initial momentum before the push.)

The two skaters pushing each other apart were a rather special case, since they were both initially at rest. It is far more common in two-body collisions that at least one, and often both, of the bodies are moving before they collide—as the two gliders did, for example. Your life is full of experiences of this kind, particularly if you play ball games, such as tennis, cricket, snooker, golf or soccer. In order to check whether you have developed some instinctive appreciation of how momentum is transferred in such situations, try the following two ITQs. (Should you feel the need for some more empirical guidance, why not try colliding two coins of unequal mass, say 10p and 1p, on a smooth and level table top?)

ITQ 1 A moving object of a small mass m collides head-on with a stationary object of a much larger mass M. Select the *two* statements from the list below, that correctly describe the motion of both objects after the collision.

A Object M—remains stationary
B —moves forward, faster than m before collision
C —moves forward, at the same velocity as m before collision
D —moves forward, more slowly than m before collision
E Object m—stops dead when it hits M
F —moves forward with M, at the same velocity
G —moves forward, more slowly than M
H —moves backwards, very fast (faster than it moved forward before)
J —moves backwards, more slowly than it moved forward before collision.

ITQ 2 Suppose now that the two objects M and m interchange their roles. That is, the heavy object M is moving and collides with the stationary object of small mass m. Formulate your anticipation of how the two bodies will be moving after the collision (use statements similar in form to A–J in ITQ 1).

Perhaps you are just beginning to feel that the conservation of momentum is all you need to be able to anticipate the outcome of any collision between material objects. Is this really the case? Well, try the next ITQ to see whether it is or not.

ITQ 3 Two gliders on a frictionless track approach each other on a head-on collision course. Both gliders have an identical size and shape and the same mass m. Both have the same speed v ($\vec{v}_1 = +v$, $\vec{v}_2 = -v$). Decide which of the following results contradict the law of conservation of momentum.

A Both gliders stop dead when they collide.
B Both gliders move in the same direction after the collision ($\vec{v}_1 = \vec{v}_2 = +v$).
C The gliders reverse their directions of motion, so that their velocities are opposite to what they were before the collision ($\vec{v}_1 = -v$, $\vec{v}_2 = +v$).
D The two gliders move in opposite directions at velocities $\vec{v}_1 = -v/2$, $\vec{v}_2 = +v/2$.
E The two gliders have velocities $\vec{v}_1 = -2v$, $\vec{v}_2 = +2v$.
F The two gliders can have *any* speed after the collision, provided it is the same for both and in opposite directions ($\vec{v}_1 = -\vec{v}_2$).

Compare your answer with your everyday experience. Is the conservation of momentum a sufficient guide to the outcome of collisions?

Clearly, the conservation of momentum alone does not give a *unique* prediction of the outcome of a collision. Without beating about the bush, let us say straight away that in order to get a unique prediction, which agrees with the actual result, you would have to take into account another conservation law—the law of *conservation of energy*, first introduced in Unit 8.

ITQ 4 Check all alternatives A to F in ITQ 3 for conservation of energy. (Assume that the collision is elastic—that no energy is lost in deformations.) Find out if there is an alternative that satisfies *both* the conservation laws for energy and for momentum. If there is one, does it agree with what in your experience will actually happen?

Thus we have reached an all-important conclusion: in order to describe the result of collisions between objects, or to anticipate the outcome of such collisions, it is necessary to apply *simultaneously* the conservation laws for *both* energy and momentum.

To gain a little bit more practice in the simultaneous application of the two conservation laws do SAQ 2 now.

SAQ 2 Consider an elastic head-on collision between two objects of the same mass m (such as two identical gliders or pucks on an air table). Initially, object 1 is moving from left to right with velocity $\vec{v} = +v$ towards object 2, which is stationary. (Note that by choosing the left-to-right direction as positive, the velocities and the momenta in the opposite direction will be negative. The use of + and − signs in this case replaces, or rather specifies the meaning of, the direction arrows.)

Table 1 lists a number of hypothetical alternatives for the motions of the two objects after the collision. Check all these alternatives for the conservation of both energy and momentum, and identify the one that correctly describes what actually happens. Symbols $\vec{p}$ and E_k stand for the total momentum and the total kinetic energy of both objects respectively; the numbers 1 and 2 identify the two objects.

TABLE 1 Hypothetical results of an elastic head-on collision between two identical objects

Before collision

①→ $\vec{v} = +v$ (m) ② $\vec{v} = 0$ (m)	$\vec{p} = m\vec{v} = +mv$ $E_k = \frac{1}{2}mv^2$

After collision

	Motion of objects 1 and 2	$\vec{p}$	E_k
A	$\vec{v}_1 = -\frac{v}{2}$ ←① ②→ $\vec{v}_2 = +\frac{v}{2}$	0	$\frac{1}{4}mv^2$
B	①②→ $\vec{v}_1 = \vec{v}_2 = +\frac{v}{2}$		
C	$\vec{v}_1 = -\frac{v}{\sqrt{2}}$ ←① ②→ $\vec{v}_2 = +\frac{v}{\sqrt{2}}$		
D	①②→ $\vec{v}_1 = \vec{v}_2 = +\frac{v}{\sqrt{2}}$		
E	$\vec{v}_1 = 0$ ① ②→ $\vec{v}_2 = +v$		
F	$\vec{v}_1 = +\frac{v}{2}$ ①→ ②→ $\vec{v}_2 = +\sqrt{3}\frac{v}{2}$		
G	$\vec{v}_1 = -\frac{v}{2}$ ←① ②→ $\vec{v}_2 = +3\frac{v}{2}$		

After you have done this SAQ, you should need no further proof that *both* conservation laws are equally important and that both must be applied simultaneously to each problem. There were several alternatives that conserved momentum (B, E, G) and several that conserved energy (C, D, E, F), but only one of them (E) conserved both. And that is, indeed, the one and only alternative that *always* happens in a collision of this type. Just think how horrifying this world of ours would be, if only one of the two laws worked! There would be no cricket, amongst other things.

There is just one more point to be made about conservation of momentum. In all the examples we have chosen so far, both objects always moved along the same straight line. So the direction of velocity and momentum was easily specified by the plus and minus signs. However, in real life objects move in different directions and, often, more than two objects are involved in collisions. The conservation of total momentum still holds, but to use it (in combination with the law of conservation of energy) for actual calculations of the velocities and momenta of all individual objects becomes a tedious exercise in number-crunching and in solving sets of simultaneous equations. Important though all this is for practical applications in physics and engineering, in the context of this Course you are expected to grasp just the basic principles. Figure 1 gives you some idea of how the problem of dealing with directions of velocities and momenta can be approached.

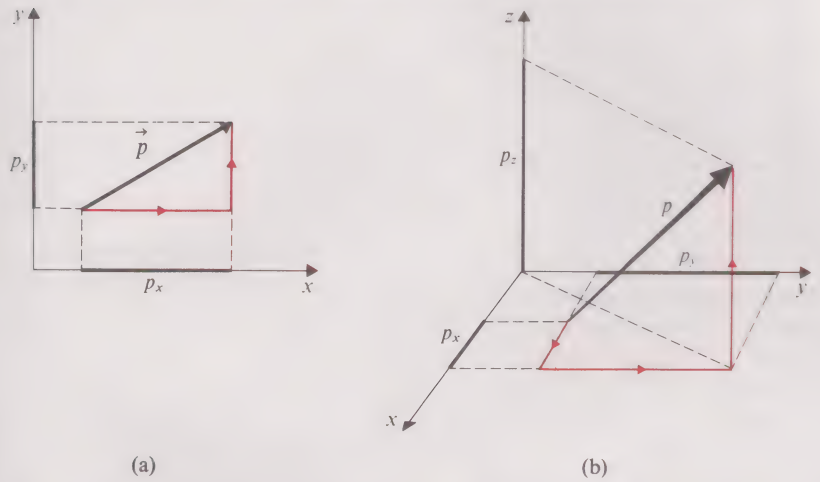


FIGURE 1 (a) Momentum in a plane, and (b) in three-dimensional space.

If you choose at random some direction in the *plane* of your desk top, you can always describe this direction with reference to *two basic directions*, perpendicular to each other. Let us say, for example, that you start from the near left corner of your desk top and choose the two basic directions (they are called axes of coordinates) along the edges of the desk top. Then the momentum $\vec{p}$ of any object moving on your desk can be represented as shown in Figure 1a. The *length* of the arrow corresponds to the magnitude of the object's momentum. The *direction* can be specified with reference to the two chosen edges of your desk (axis x—positive direction from the near left to the near right corner; axis y—positive direction from the near left to the far left corner). The arrow in Figure 1a goes upwards *and* to the right. So, you might say that the momentum this arrow represents is *composed* of a little bit of momentum upwards (along the y-axis) and another bit of momentum from left to right (along the x-axis). It is customary to call the two segments p_x and p_y, shown on the x- and y-axes, the *components* of the original momentum $\vec{p}$. If the bodies taking part in interactions can move anywhere in space, you would need three axes of coordinates (x, y, z), and each velocity and momentum would then be specified by three components (Figure 1b).

components of momentum

The most important point to remember is that if the moving objects are not confined to one straight line, but can move anywhere within a plane or in space, the law of conservation of total momentum *cannot* be expressed by a *single* formula (except in the language of vector calculus, which is not our aim here). It can be shown that if momenta are represented by their two or three components as in Figure 1, then *the conservation law holds for each component separately*. In other words, if the x-components of the momenta of all objects before collision added up to a certain value, then the sum of all the x-components after the collision *must* be the same. Since this is also true for any other axis of coordinates, the law of conservation of total momentum breaks down into two or three *separate equations*, one for each reference direction (axis of coordinates).

1.2 Beams of electrons: do electrons have a wavelength?

In the earlier Units we referred to the electron as a 'particle'. This probably led you to imagine that electrons move in space and time in very much the same way as ordinary marbles, or pellets from a shot-gun. This later association may well be reinforced by the technical term 'electron gun', used to describe a device for producing a narrow beam of electrons. As you can see in Figure 2, an electron gun consists of a hot wire filament F (heating the wire gives some electrons enough energy to escape from the surface) and a metal plate M that is positively charged in order to produce a very strong attractive force on the negatively charged electrons. The attractive force accelerates the electrons released from the filament, but only a narrow beam is allowed to pass through the small hole in the plate M. This narrow beam can subsequently be manipulated for different purposes—straightforward further acceleration by electric fields, or bending, scanning and focusing by a combination of electric and magnetic fields. Such a beam finds its uses in accelerators, X-ray machines, oscilloscopes, television tubes and electron microscopes. In all of these applications, the space within which electrons travel must be rid of air or any other gas, to prevent losses of energy and changes of direction caused by the collisions of the electrons with gas molecules.

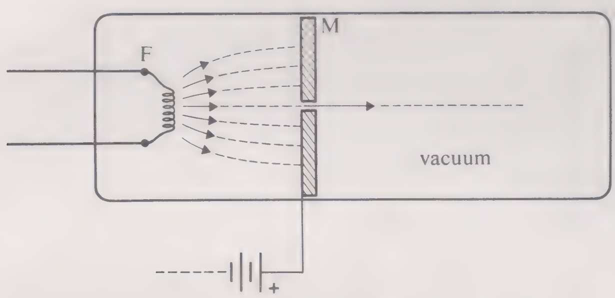


FIGURE 2 An electron gun.

Let us now return to our main problem. Is it always reasonable to think of a beam of electrons, produced by an electron gun, as a stream of little marbles? After all, you saw in Unit 9 that although light could most often be thought of as a wave motion, sometimes it had to be described as particles. What if we turn this argument round and ask whether these 'particles' that we call electrons could sometimes be described as having a wavelength? There is only one scientific way in which to answer this question, and that is to find out by experiment.

You will recall from Unit 9 that many interesting wave properties of electromagnetic radiation were revealed by shining monochromatic light through a barrier with narrow slits. Perhaps it might be revealing to perform a similar experiment with electrons. It is not very difficult to produce a narrow beam of electrons, all accelerated to the same velocity $\vec{v}$, to shine this beam onto a barrier with some suitable, regularly spaced slits and to detect the electrons after the transmission through the slits by a photographic plate on which each of the electrons will produce a visible mark.

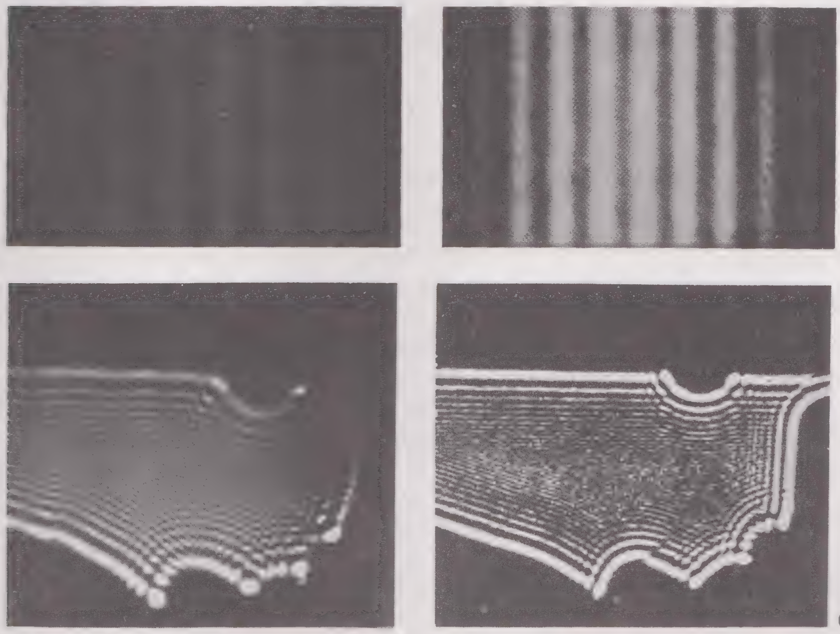
If you were to use the same slits or diffraction gratings as for light (Unit 9), the image on the photographic plate, showing the distribution of electrons after the transmission, would not be significantly different in either size or shape from the image caused by the straight electron beam. But things look very different when a beam of electrons is shone at the surface of a *perfect crystal*, in which regularly spaced gaps between atoms perform the same function as slits of a diffraction grating, only with a much *smaller separation d* than in any man-made gratings. In this case, the distribution of electrons after scattering from the crystal is widely spread and shows maxima and minima, very similar to those in the diffraction pattern of light (Unit 9).

Experiments of this kind were first done by Davisson and Germer in the U.S.A., and independently by G. P. Thomson in England (1927). They showed quite convincingly that *the angular distribution of scattered electrons can only be explained by some kind of a wave property, associated with the motion of electrons.*

An aside

Perhaps you are worried that we are comparing two different things. Diffraction patterns produced by light (Unit 9 and the Summer School experiment) were observed after the light was *transmitted* through a grating, whereas, with the elec-

trons, we are talking about diffraction patterns after the electron beam was *scattered* by a crystal. It is an experimental fact (that can be explained by the theory of wave motion) that diffraction patterns observed after scattering are of the same type as diffraction patterns after transmission. Although the first experiments with electron diffraction were done by scattering, it is possible to provide direct evidence of electron diffraction after transmission. Figure 3 compares the transmission patterns of electrons and of light, and leaves no room for doubt about the essential similarity between the two.



(a)

(b)

FIGURE 3 Diffraction of electrons and diffraction of light:

(a) diffraction patterns from a double slit—electrons on the left (C. Jönsson, University of Tübingen); light on the right.

(b) (*left*) diffraction pattern produced by electrons emitted from a sharp tip and passed through a very narrow gap between two opaque crystals (by courtesy of the Cavendish Laboratory) for comparison with (*right*) light diffracted through an opening cut in a metal plate.

Figure 4 shows in a diagrammatic form what happens when a beam of electrons is scattered at the surface of a crystal. Remember that all electrons in the beam have the same velocity $\vec{v}$ and hence *the same momentum* $\vec{p}$. The fact that they produce a diffraction pattern after scattering means that, whatever wave it is that is associated with the motion of the beam, it must have a *well-defined wavelength* λ. (You will recall that only monochromatic light produces a sharp diffraction pattern.) Thus it is reasonable to expect that the wavelength of a wave associated with the motion of the electrons is related to their momentum. This can be checked easily

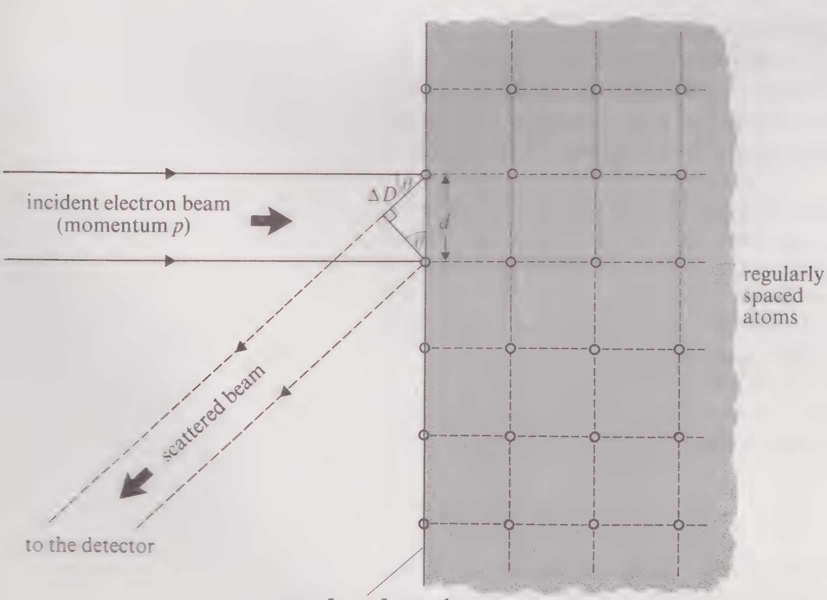


FIGURE 4 Scattering of electrons at the surface of a crystal.

by repeating the same experiment with beams of different momentum. If the momentum and wavelength are linked, then a different momentum must lead to a different spacing in the diffraction pattern. This is indeed the case.

In Unit 9 (and in the Summer School experiment) you were able to calculate the wavelength of light from the measured angular separation between the intensity maxima in the diffraction pattern. Since the similarity of diffraction patterns of light and electrons has already been established, we can use the same approach here. Looking at Figure 4, you can expect that there will be a maximum of electron intensity at an angle θ, provided the path difference ΔD is equal to a whole number of wavelengths:

$$\Delta D = \lambda, 2\lambda, 3\lambda, \ldots \tag{2}$$

The path difference ΔD can be expressed in terms of the regular spacing d between the atoms in the crystal:

$$\Delta D = d \sin \theta \tag{3}$$

The first maximum will be observed at an angle θ_1 for which ΔD_1 is equal to one wavelength, the second at θ_2 for which $\Delta D_2 = 2\lambda$ and so on. Hence you can write (just as you did for light):

$$d \sin \theta_n = n\lambda \tag{4}$$

where θ_n is the angle at which the nth maximum is observed.

The separation between atoms d can be determined for each crystal by independent experiments. So, by measuring the angles of diffraction maxima, the wavelength λ associated with the beam of electrons of a particular momentum $\vec{p}$ can be calculated*. The result is that the wavelength λ and the magnitude of the momentum p are related in the following way:

de Broglie formula

$$\lambda = \frac{h}{p} \tag{5}$$

where $h = 6.63 \times 10^{-34}$ J s is Planck's constant, which we first mentioned in Unit 9 in connection with the photoelectric effect.

Comment In formula 5 we have dropped the arrow indicating the direction of momentum. Strictly speaking, the relationship between λ and p should be written so as to show that the wave associated with the beam travels in the same direction as the beam. But, for most practical purposes, it is the relationship between the wavelength and the magnitude of the momentum that is important.

Formula 5 is exceedingly important, because it firmly links one parameter of the wave-like model for the motion of an electron beam (the wavelength λ) to one parameter of the particle-like model of the electrons (the magnitude of the momentum p). The fact that the two parameters are linked together through Planck's constant is not an accident.

E=hf

ITQ 5 Do you recall the relationship between the frequency f of monochromatic light and the energy E carried by the photons of this light, when it interacted with the electrons in metals (photoelectric effect) or with the electrons confined within atoms (absorption and emission line spectra)? What is the essential similarity between that relationship and formula 5?

Although formula 5 has been presented here as a result of experimental observations, it is interesting to note that it was first suggested in 1924 by the French theoretical physicist, Louis de Broglie. At that time, any suggestion that beams of electrons may behave in some circumstances as waves seemed crazy to many people. But de Broglie argued that if light needs two different models for the proper description of all observed effects, perhaps particles too may show some similar dual behaviour hitherto unsuspected and unnoticed? It took three years before experimentalists proved that he was right. Formula 5 is, therefore, generally known as the *de Broglie formula* and the wavelength associated with the momentum of

* Figure 4 implies that electrons are scattered only by the first layer of atoms. In reality, some electrons penetrate deeper into the crystal and are scattered by subsequent atomic layers. Also, the arrangement of atoms within the crystals is often more complicated than just straight rows or columns. All this makes the diffraction pattern more complex, but as long as there is regularity in the structure of the crystal, and provided the beam has a well defined momentum, it is always possible to analyse the diffraction pattern mathematically to obtain a numerical value for λ.

moving particles as the de Broglie wavelength. In order to avoid confusion with the wavelength of light, sound, radio waves or water waves, we shall use a subscript dB for the de Broglie wavelength:

de Broglie wavelength $\lambda_{dB} = h/p$

$$\boxed{\lambda_{dB} = \frac{h}{p}} \qquad (5)$$

SAQ 3 A beam of electrons is accelerated to the speed of $10^5\,\mathrm{m\,s^{-1}}$. Taking the mass of an electron approximately as $10^{-30}\,\mathrm{kg}$ and $h \approx 10^{-33}\,\mathrm{J\,s}$, calculate the order of magnitude of the de Broglie wavelength associated with such electrons. Compare it with the wavelength of visible light and explain why an ordinary diffraction grating of $d \approx 10^{-6}\,\mathrm{m}$ cannot produce observable diffraction patterns with electrons.

1.3 Beams of heavier particles

Having seen that electrons are propagated in the same way as waves, we now move on to other types of radiation, to beams of considerably heavier particles. As you will recall from Units 10 and 11, most of the mass of an atom is contained in its nucleus. Roughly speaking, an electron is about 2 000 times lighter than a proton or a neutron. Thus, at the same velocity, the momentum of a proton is about 2 000 times larger than that of an electron and, correspondingly, its de Broglie wavelength 2 000 times smaller. Equation 4 tells you that if λ_{dB} is 2 000 times smaller, so will $\sin\theta$ be (for the same d). This means that the angular separation between diffraction maxima for protons and neutrons scattered by crystals would be very small indeed. In practice, clear diffraction patterns can only be obtained for *very slow* neutrons and protons, when the wavelength is relatively long. (Incidentally, the diffraction of slow neutrons is one of the most important modern techniques for studying the structure of crystals.)

For faster protons and neutrons, and for beams of light nuclei, the diffraction effects can be observed in collisions with other nuclei, atoms or molecules. Thus, for example, Figure 5 shows how many *deuterons* (a deuteron is a nucleus consisting of one proton and one neutron) are observed in different directions, with respect to the direction of the original beam, after the deuterons have collided with the nuclei of nitrogen. After the collision, most deuterons are propagated more or less forward in the initial direction (around 0°), but there are two clearly visible maxima at larger angles. This curve shows a remarkable similarity with the distribution of light diffracted through a slit, as discussed in Unit 9. Compare the shape of the shaded half of Figure 6 with the shape of the curve in Figure 5.

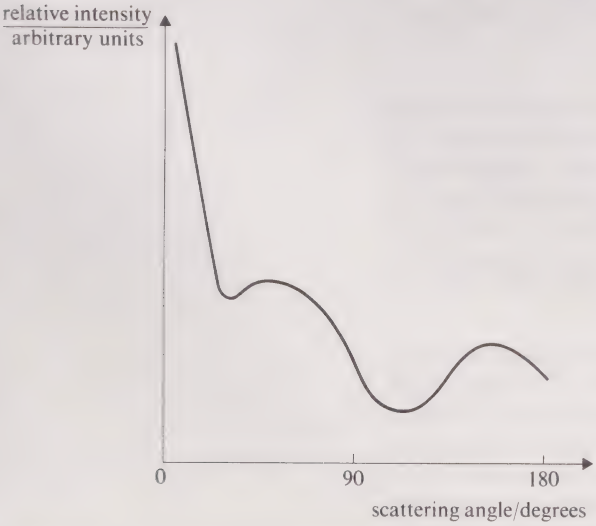


FIGURE 5 Angular distribution of intensity of deuterons elastically scattered by nitrogen (adapted from results obtained by W. M. Gibson and E. E. Thomas).

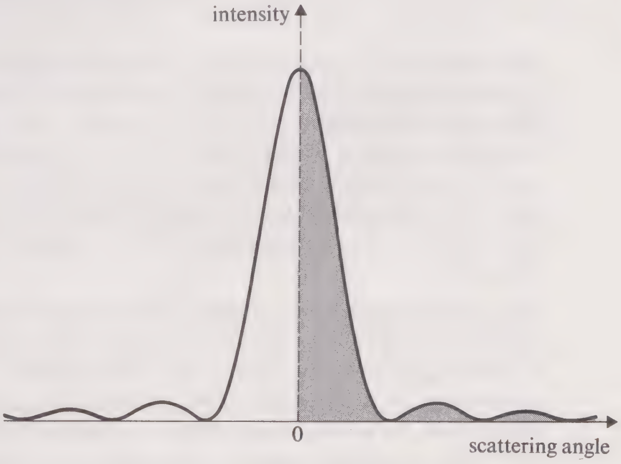


FIGURE 6 Distribution of the intensity of light diffracted in passing through a slit, or in scattering at a small obstacle.

Similar diffraction phenomena have now been observed for beams of *heavier nuclei*, as well as for beams of *atoms and molecules*. This is very significant, in that whereas electrons, protons and neutrons can be regarded for all practical purposes as 'elementary particles', atoms and molecules are clearly groups of many different particles bound together. Nevertheless, in collisions with other target atoms, they are diffracted *as complete units*. Moreover, the angles of the diffraction maxima of these large particles depend upon their momentum in exact accordance with the de Broglie formula, just as they did for the electrons.

It is, therefore, established experimentally that *all* beams of particles—from electrons, protons and neutrons, to nuclei, atoms and molecules—behave in some way as a wavelike motion. The characteristic wavelengths of their waves are determined by their momenta, according to the de Broglie formula 5.

1.4 Propagation of macroscopic objects

In the previous Sections we have been concerned with particles so small that most of them, with the exception of large molecules, are beyond the range of direct visual observations. You might well think that although it sounds odd to talk about the beams of such particles being propagated as waves, it has to be accepted on the strength of the experimental evidence. But surely it would be outrageous to suggest that streams of spherical marbles, billiard balls, or bullets fired from a machine-gun, are propagated as waves?

Clearly, if you were to aim a stream of marbles of diameter 8 mm at the centre of a gap of, say, 10 mm width in a wall, you would not see any diffraction pattern at the other side. Each marble would just go straight through. Obviously, the idea of wavelike motion does not apply to such objects. Or does it?

This argument may sound very convincing, but it contains a major flaw. Can you see what it is? (*Hint* Recall the condition for the observation of diffraction effects (SAQ 3).)

Diffraction effects are determined by the relative size of the *wavelength* and the size of the slit. Hence the argument above, which is based on the *physical size* of the marbles, is irrelevant to the wavelike properties of a stream of marbles.

> **SAQ 4** If a marble of mass 10^{-2} kg, moving with speed $1\ \text{m s}^{-1}$, had a wave-like nature, what would be its de Broglie wavelength λ_{dB}? Could any diffraction effects be observed when a stream of such marbles pass through a gap 10 mm wide?
>
> 1 mm = 10^{-3} m
>
> **SAQ 5** Suppose that a stream of marbles, such as that described in SAQ 4, is made to pass through a grid (diffraction grating) containing gaps (each of which is sufficiently large to let a marble through), regularly spaced with the separation $d = 10^{-2}$ m. Calculate the angle at which the first diffraction maximum would occur.

Your answers to SAQs 4 and 5 tell you that for objects of everyday mass, moving at everyday speeds, the de Broglie wavelength is so minute that corresponding diffraction effects cannot be observed. But this is not the same as saying that the de Broglie relation is wrong, or that it is not universally applicable. Although you cannot experimentally *prove* the existence of diffraction effects for marbles and balls, the application of the de Broglie formula does not lead to any contradictions. Indeed, its consequences are in complete agreement with ordinary observations.

Thus the non-observation of diffraction effects for marbles confirms, rather than rejects, the claim that the *wave theory applies to the propagation of all objects*, large or small, light or heavy, slow or fast. This enables us to use the same kind of mathematics to describe the motion of electrons, as well as that of marbles and billiard balls. As you know from Unit 3, material objects not acted upon by any force move in straight lines. In the context of the wave theory of motion, the *motion of a body in a straight line* can be regarded as a limiting case, as a wave motion in which the wave has a *negligibly small wavelength*.

A brief summary of Sections 1.2 to 1.4

All objects, normally regarded as particles, display some wave properties. These are manifest in diffraction effects. Mathematically all such diffraction effects can be described by associating with the momentum $\vec{p}$ of the particle a ***de Broglie wavelength*** λ_{dB}, according to the formula

$$\lambda_{dB} = \frac{h}{p}$$

where $h = 6.63 \times 10^{-34}$ J s and is Planck's constant.

The de Broglie wavelengths of common-or-garden objects of everyday life are too small to produce observable diffraction effects. Their motion in a straight line can be regarded as wave motion in which the wave has a negligibly small wavelength.

1.5 The principles governing the propagation of waves and particles

Section 1.5 can be treated as *optional reading.* It describes an interesting aspect of the underlying uniformity of nature, but it is not absolutely essential to the main story of the Unit.

In Unit 4 you met a very general principle that helped you to understand how seismic waves are propagated through the Earth. The same principle, known as the *Fermat's principle of least time*, also applies to other kinds of wave motion, such as sound, light and radio waves. Its applicability to light is nicely demonstrated in Figure 7. The light from the Canigou mountain could not have reached the photographer's camera in Marseilles by travelling in a straight line, since a straight line connecting the two places goes through the land and water, impenetrable by light. The light must have followed a curved path, determined by the variations in the density of the air above the Earth's surface between the two places. The variations of density lead to the variations of the velocity of light in different layers of the atmosphere, with the result that the actual path of light (the quickest one) is not the one along the straight line. For all kinds of wave motion, knowledge of the changes in the medium makes it possible to predict the path the wave disturbance will follow. And vice versa, from the observation of the path of the wave motion, it is possible to obtain some information about the changes in the properties of the medium.

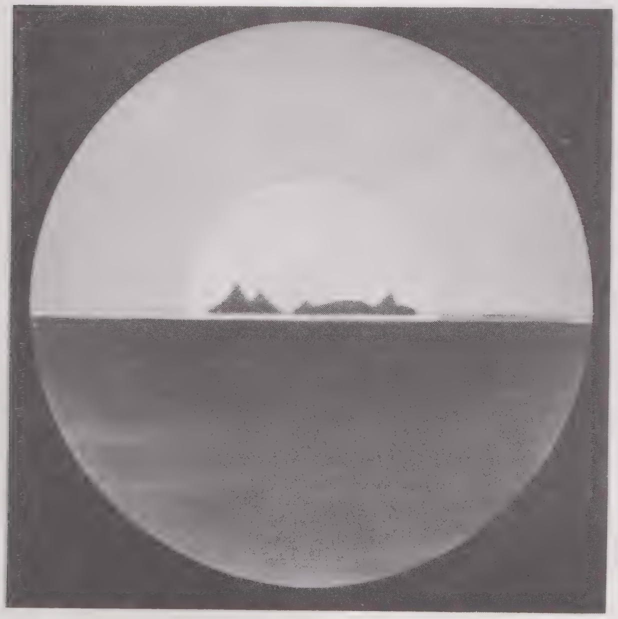


FIGURE 7 The Canigou Mountain, seen from Marseilles, silhouetted against the setting Sun. The mountain lies below the horizon of the observer at Marseilles and can only be seen because light follows a curved path.

In the previous Sections of this text, you have been presented with experimental evidence that beams of particles exhibit diffraction effects, thus indicating some kind of wave-like behaviour. You may well ask: does Fermat's principle of least time apply also to the motion of electrons, atoms, marbles and stones?

Suppose that all you know is that a stone thrown from a point A has landed at a point B on the Earth's surface. Will Fermat' s principle enable you to choose the path along which the stone travelled between A and B? (See Figure 8.) The question cannot be answered, unless some information is given about the initial conditions, such as the mass of the stone and the velocity given to it at point A (note that velocity means both the direction of the throw and the speed). After all, even for light, Fermat's principle can be usefully applied only when its frequency and its velocities in the different media are known.

Let us analyse the situation when the stone is thrown from A with the same initial speed, but at different angles. Which of the paths, shown in Figure 8, would be selected by the straightforward application of the Fermat's principle?

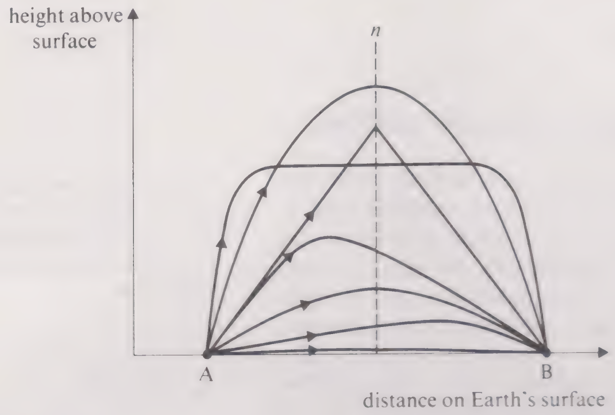


FIGURE 8 Hypothetical paths of a stone thrown from A to B.

Given the same initial speed, it would appear that the quickest route from A to B would be along the horizontal straight line. Every other path is longer and is also followed at a lower average speed, as the stone always slows down on the way up.

> Can a stone from A reach B along a horizontal straight line?
>
> Excluding the unreal cases of either an infinitely short distance between A and B or an infinitely fast speed, clearly not. The stone will be pulled down along the way from A to B by the force of gravity.

Thus a direct application of Fermat's principle leads to the choice of a path that is physically impossible.

Of course, it would be possible to analyse our problem by the application of Newton's laws, the theory of gravitation and the conservation laws for energy and momentum. This would show that, for a given distance A–B, a given mass m and a given direction of throw, there is one and only one value for the initial speed at which the stone will reach B, along a unique path, that is of parabolic shape, symmetrical about the line n (Figure 8).

Is there a simple principle that would single out this unique path in a similar way to that in which Fermat's principle of least time singles out the path of a wave motion? Well, it turns out that there is such a principle and that it was first formulated more than two centuries ago by a French physicist by the name of Maupertuis (1699–1759). In order to appreciate the meaning of this principle, it is necessary to introduce a new physical quantity, which you have not met before.

Figure 9 shows the path of a particle (a stone, a gun shot or any other projectile) divided into small segments of equal length Δs. The size of the segments is chosen so that within each one of them the momentum $\vec{p}$ of the moving projectile can be considered approximately constant. (Clearly, for maximum accuracy the length of the segments Δs must be as small as possible. In mathematical language we would say that 'Δs should converge to a point'.) Since the momentum changes continuously all the way from A to B, the product of $p \cdot \Delta s$ will have a different value for each individual segment. For the whole path we can now define a new quantity, called *action*, as follows: **action**

$$a(\mathrm{A} \rightarrow \mathrm{B}) = p_1\Delta s + p_2\Delta s + p_3\Delta s + \cdots \qquad (6)$$

Or, in a shorter form:

$$a(\mathrm{A} \rightarrow \mathrm{B}) = \sum_{\mathrm{A}}^{\mathrm{B}} \Delta s \cdot p_H, \tag{7}$$

where the symbol $\sum_{\mathrm{A}}^{\mathrm{B}}$ stands for the *sum* of all contributions $p \cdot \Delta s$ from individual segments all the way from A to B and p_H stands for the magnitude of the momentum, which clearly depends on the height H of each segment above the level A–B.

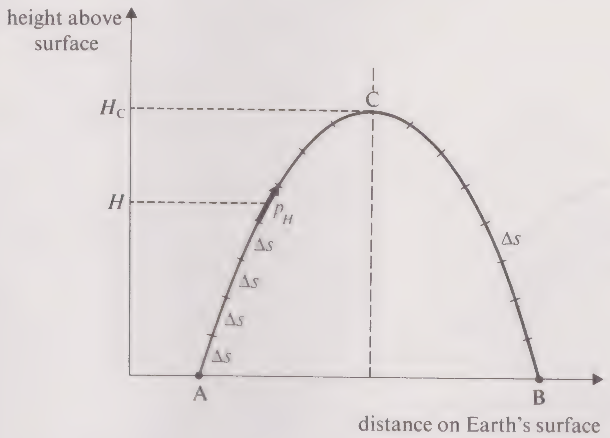


FIGURE 9 Maupertuis's principle of least action (motion of a particle).

An aside

It is easy to derive an explicit formula for the magnitude of the momentum p_H as a function of height H, provided the initial kinetic energy E at point A is known. From the conservation of energy (Unit 8) it follows that, for any point at height H, it must be true that:

$$E = \tfrac{1}{2}mv_H^2 + mgH \tag{8}$$

where v_H is the speed of the projectile at the height H and g is the acceleration due to gravity. Bring the term mgH to the left-hand side of equation 8 and multiply both sides by $2m$:

$$2m(E - mgH) = m^2v_H^2 = p_H^2$$

and hence:

$$p_H = \sqrt{2m(E - mgH)} \tag{9}$$

Maupertuis' principle (of least action)

Maupertuis's principle of least action, which singles out the one and only path that a particle of mass m and initial energy E will follow between A and B, says that the total action along the path must be a minimum. That is, using expression 7:

$$a(\mathrm{A} \rightarrow \mathrm{B}) = \sum_{\mathrm{A}}^{\mathrm{B}} \Delta s \cdot p_H \text{ is a minimum} \tag{10}$$

It is encouraging to note some basic similarity in the formulation of this principle and Fermat's principle. In both cases, the path of propagation seems to be determined by the minimum possible value of one physical quantity—time for waves, action for particles. Indeed, the similarity between the two principles is even closer. It is possible to reformulate Fermat's principle of least time in a way analogous to that in expression 10.

In Figure 10, the path of light from A to B is divided into small segments of equal length Δs. Since the velocity of light depends on the density of the atmosphere, which in turn varies with height H, it is desirable to choose the length of the segments so that within each one of them the velocity can be regarded as approximately constant. Each segment can then be characterized by the time Δt_H it takes light to travel through it, that is:

$$\Delta t_H = \frac{\Delta s}{v_H} \tag{11}$$

The time taken over the whole route from A to B can then be expressed as:

$$t(\mathrm{A} \rightarrow \mathrm{B}) = \sum_{\mathrm{A}}^{\mathrm{B}} \Delta t_H = \sum_{\mathrm{A}}^{\mathrm{B}} \frac{\Delta s}{v_H} \tag{12}$$

where the summation sign $\sum_{\mathrm{A}}^{\mathrm{B}}$ and the subscript H have the same meanings as before (expression 7). If the velocity of light is known as a function of height, the

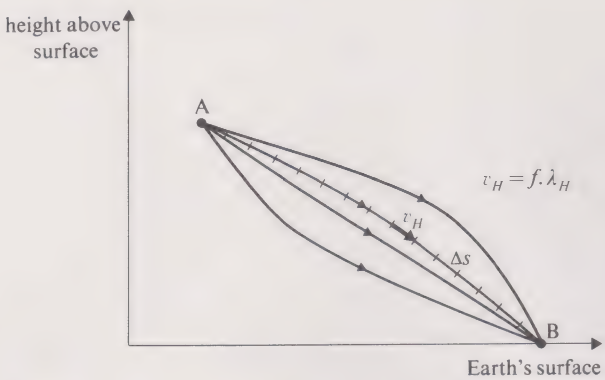


FIGURE 10 Fermat's principle of least time (propagation of light).

value of expression 12 can be calculated for any conceivable route from A to B (Figure 10), and the path that the light will actually take will be the one for which:

$$t(\mathrm{A} \rightarrow \mathrm{B}) = \sum_{\mathrm{A}}^{\mathrm{B}} \frac{\Delta s}{v_H} \text{ is a minimum} \tag{13}$$

This is very similar in form to Maupertuis's principle (expression 10), though clearly not identical. It should not really be all that surprising to find out that Fermat's principle does not apply to the motion of particles in the same way as it does to wave motion. After all, for every kind of wave motion that you have met before, its propagation through a medium did not involve any bulk transfer of the mass of that medium in the direction of the wave. It was the *disturbance* of the medium that travelled forward, not the medium itself. On the other hand, in a beam of electrons or in a volley of gunshots, it is the electrons and the bullets that travel—there is a real *transfer of mass* forward, in the direction of motion. Thus the waves associated with beams of particles (de Broglie waves) must be of a *different nature* from that of any other waves. We shall return to this in Section 3.

However, in spite of their different physical nature, *all* wave motions can be described by the same type of mathematics. Thus, if you substitute into the formulation of Maupertuis's principle of least action, as given in expression 10, the de Broglie formula for the wavelength λ_{dB} associated with particles of momentum $\vec{p}$, you get:

$$a(\mathrm{A} \rightarrow \mathrm{B}) = \sum_{\mathrm{A}}^{\mathrm{B}} \Delta s \left(\frac{h}{\lambda_{\mathrm{dB}}} \right) \tag{14}$$

where h is Planck's constant. Remembering that h is a universal constant, the requirement of Maupertuis's principle can be expressed thus:

$$\begin{gathered} \sum_{\mathrm{A}}^{\mathrm{B}} \frac{\Delta s}{\lambda_{\mathrm{dB}}} \textit{ must be a minimum} \\ \textit{for the path of particles } \left(\lambda_{\mathrm{dB}} = \frac{h}{p} \right) \end{gathered} \tag{15}$$

Similarly, for any usual wave motion, the velocity v_H in expression 13 can be replaced by the product $f \cdot \lambda_H$, where the frequency f is a constant for the given

radiation (Unit 9) and λ_H changes with the changes of v_H. Thus, expression 13 can be written as:

$$t(\mathrm{A} \to \mathrm{B}) = \sum_{\mathrm{A}}^{\mathrm{B}} \frac{\Delta s}{f \cdot \lambda_H} \tag{16}$$

and Fermat's principle then requires that:

$$\sum_{\mathrm{A}}^{\mathrm{B}} \frac{\Delta s}{\lambda_H} \textit{ must be a minimum}$$
$$\textit{for the path of wave motion } \left(\lambda_H = \frac{v_H}{f}\right) \tag{17}$$

Note that statements 15 and 17 are of identical form.

Although we should always be aware of the different *nature* of the waves, it is possible to formulate a simple, general, principle of propagation, covering all kinds of radiation:

> *Any type of radiation chooses that path between two points that is characterized by the least number of wavelengths.*

ITQ 6 Explain why statements 15 and 17 are equivalent to the requirement of the least number of wavelengths.

1.6 Wave trains and wave packets

If this idea of describing the propagation of objects in terms of wave motion is new to you, as it probably is, you may well find it takes a time to assimilate. For example, you may want to argue that macroscopic objects do not *look* like waves. Travelling waves spread out through space, but moving particles tend to be more localized, they have well-defined shapes and positions.

If this is what you are thinking, it would help you to recall that there are different kinds of travelling waves. On the one hand, there are travelling waves consisting of a continuous sequence of regularly spaced ups and downs (crests and troughs). This regular spacing defines the period (Unit 1) of such a *wave train*. Ripples on the water surface, spreading away from the point of disturbance, are one example. Sound spreading out from the vibrating part of a musical instrument or a tuning fork is another example of a wave train, although here, instead of ups and downs, we have regularly spaced regions of compressed and rarified air.

travelling wave trains

On the other hand, if you pick up one end of a long, loose rope and give it a sharp jerk upwards, you will create a single hump, travelling forward along the rope. This is also a travelling wave, because all parts of the medium (the rope in this case) experience this up-and-down wobble, albeit not all at the same time.

Which type of wave motion do you think is more likely to be associated with the motion of a macroscopic object?

As we have said, macroscopic objects normally have well-specified positions in space. If one wants to use a wave model to specify the position of a particle, it is natural to choose wave motion that, at a given time, is also more or less localized. So, one would prefer the single-jerk-type travelling wave to a continuous wave train. But there is a snag.

If you accept a travelling hump as a model for a moving particle, what would you associate with the de Broglie wavelength of the object?

We seem to have a problem. Wavelength and frequency have well-defined meanings for a travelling wave train. But how can you speak of a wavelength in the case of a travelling hump of, perhaps, irregular shape?

Fortunately, there is a way out of this tight corner. It turns out that a travelling hump of almost any shape can be regarded as *a combination, a superposition, of a number of infinitely long wave trains.* It is, therefore, customary to call such a less regular, single-hump-type travelling disturbance *a wave packet*, indicating that it is created by packing together many individual wave trains of different frequencies. To make this idea clearer and more acceptable, look at Figure 11. The two travelling wave trains A and B have a different wavelength. They are out of step at the origin O (one going up, the other down) but, because of their different wavelengths, they get repeatedly in step and out of step later. The disturbance that results from their combination (superposition) has the form of widely spaced travelling humps, with zero amplitude at X and maximum amplitude at Y.

travelling wave packets

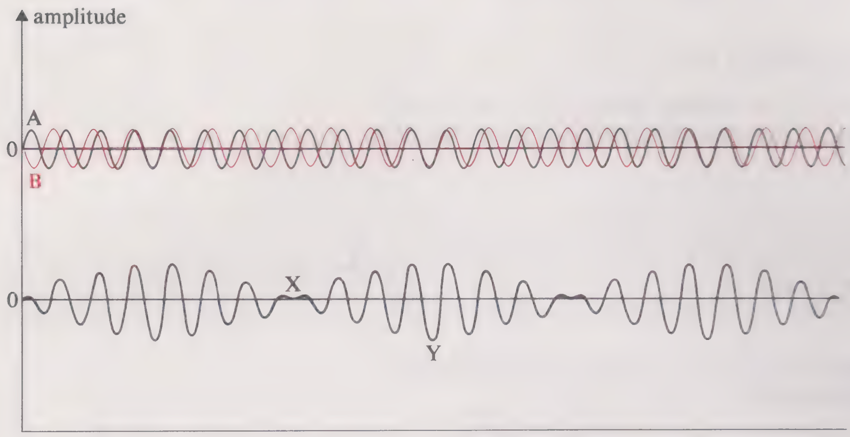


FIGURE 11 Two wave trains A and B of different frequency combine to produce a disturbance consisting of widely spaced humps.

As you can see, the combination of just two wave trains already creates a series of wave packets. By choosing a different pair of waves, we can produce wave packets of different length and separation. It is not difficult to believe that, by adding a third wave train of suitably chosen wavelength, we could further enhance one particular wave packet and suppress the neighbouring ones. By a judicious combination of a larger number of wave trains with suitably chosen wavelengths and amplitudes we can eventually create a single travelling wave packet of any desirable shape and size. Conversely, a wave packet of given shape and size can be analysed and decomposed, as it were, into its component wave trains. There is a special mathematical procedure (called Fourier analysis) for this purpose, and there are electronic instruments that provide a visual display of wave trains and wave packets. Figure 12 gives you an indication of this. Pictures A to K show ten different wave trains, and the other five pictures show the results of combining different component wave trains.

Notice how combining more wave trains leads to a narrower, more localized, wave packet. Thus the narrower the wave packet, the more frequencies are involved in its composition and the less meaningful it is to speak about the packet's frequency or wavelength. *Wave packets cannot have a precisely defined wavelength*, they are mixtures of wave trains of different wavelengths.

To recapitulate:

we know that moving particles have some kind of wave properties;

we find that the motion of a particle can be modelled by a travelling wave packet;

in constructing a travelling wave packet from a large number of different wave trains we *lose* the ability to describe it in terms of wavelength or frequency.

The *narrower* the wave packet, the better it resembles a moving particle. The position of such a narrow wave packet at a given instant can be a true representation of the *position* of a moving particle at the same instant. However, there is a price to be paid for this. As you know from Section 1.2, there is a direct relationship between the momentum of moving particles and the wavelength that corresponds to their motion and determines their diffraction patterns. A narrow wave packet, as you have just seen, does not have a well-defined wavelength and, therefore, it cannot convey any information about the *momentum* of the moving particle that it represents. Clearly, you cannot have it both ways, when you have to take into account both the particle and the wave nature of matter. The more you know about the position (narrow wave packet), the less you can tell about the momentum of a particle (the wavelength of the wave packet is not defined). We shall return to this problem later (Section 4).

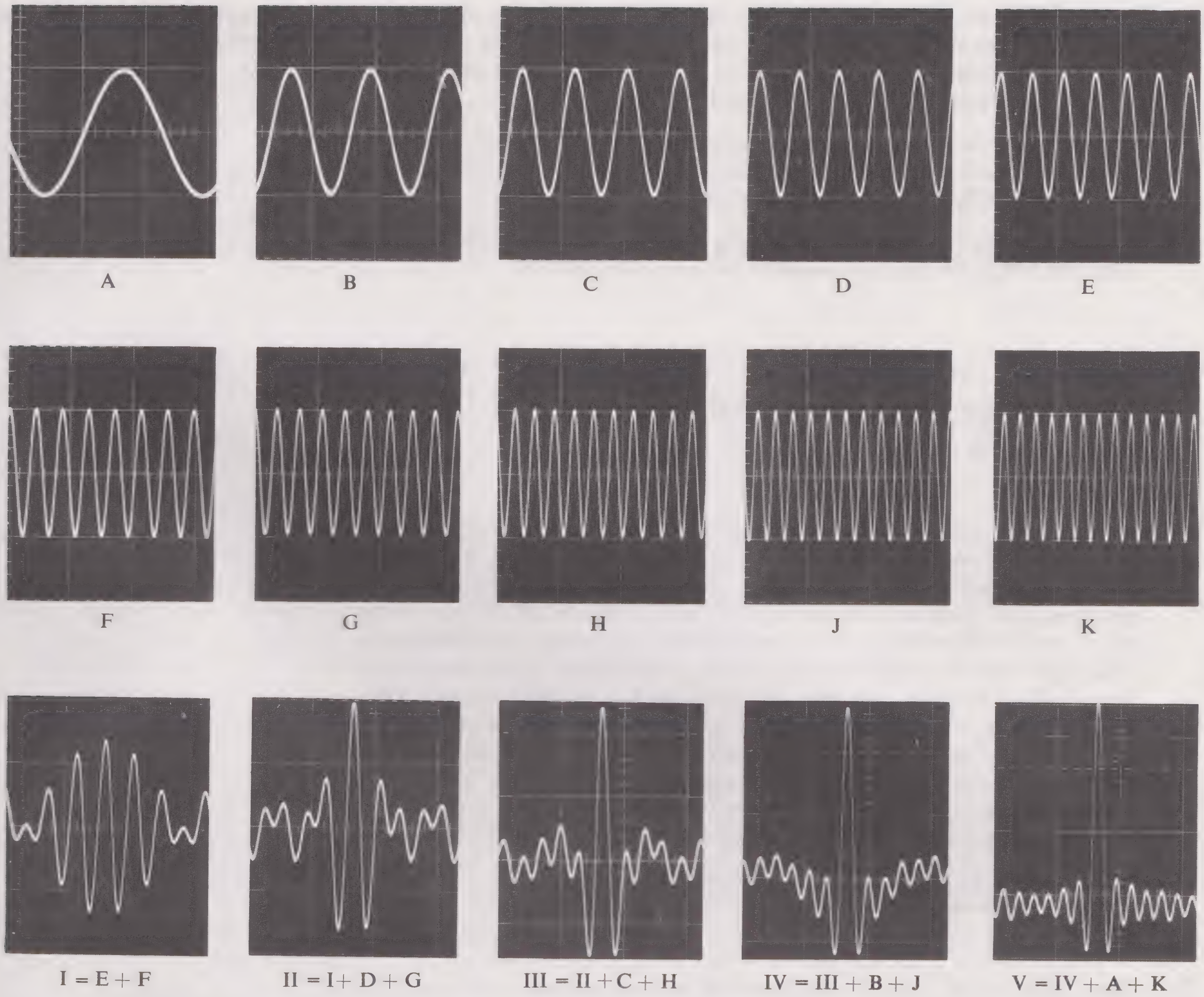


FIGURE 12 Ten wave trains of different wavelength (A to K) and five wavepackets (I to V) combined from different wavelengths (display on the screen of an oscilloscope).

Before we leave the subject of waves, note that in this Unit we are dealing with continuous *travelling* wave trains and with *travelling* wave packets. These are suitable models for particles *moving in space*. But what about particles that are *confined* within small closed systems, such as, for example, electrons in atoms? We shall answer this question in Unit 30.

1.7 Summary of Section 1

Before you move on, here is a summary of what you should have learnt from this Section:

1 In any collisions between particles, the total momentum is always conserved; total energy is also always conserved (if there is no friction and no permanent deformations). Simultaneous application of both conservation laws makes it possible to predict unambiguously the outcome of such interactions. (Section 1.1; ITQs 1–4; SAQs 1 and 2)

2 Electrons, protons, neutrons, nuclei, atoms and molecules exhibit observable diffraction effects (Sections 1.2, 1.3).

3 The diffraction patterns produced by beams of particles can be described by associating with each particle of momentum $\vec{p}$ a de Broglie wavelength $\lambda_{\text{dB}} = h/p$. (Sections 1.2 and 1.3; SAQ 3; ITQ 5)

4 The de Broglie formula can be applied to particles of any size, including macroscopic objects. However, for particles of large mass (large momentum), the de Broglie wavelength is so extremely small that associated diffraction effects cannot be observed. (Section 1.4; SAQs 4 and 5)

5 The principles that govern the propagation of electromagnetic radiation (Fermat) and the motion of particles (Maupertuis) have been shown to have the same mathematical form. Thus, all forms of propagation of energy and momentum can be described by wave theory. (Section 1.5; ITQ 6)

propagation of any radiation is described by waves

6 The position of a body moving in space can be modelled by a travelling wave packet. Wave packets do not have a precise wavelength; they are combined from wave trains of different wavelengths. (Section 1.6)

If all items in this Summary mean something to you and if you were able to do the ITQs and SAQs, you will have achieved the aim of this Section.

2 How energy and momentum are transferred in interactions

Section 1 dealt with the description of motion. The main result is that a unifying mathematical model has been found that can describe equally well the *propagation* of energy and momentum, whether carried in the form of seismic waves, electromagnetic radiation, or beams of particles. You may recall that in Unit 9 the wave model of light was found to be appropriate for the explanation of all effects in which light simply travelled from one medium to another or was transmitted through slits in barriers. In Section 1 of this text, we have extended the application of a modified wave model to the propagation of atomic particles and macroscopic objects. But in Unit 9 the wave model was found to be inadequate for the understanding of processes such as those involved in the *transfer* of energy from light to electrons. We had to introduce the concept of photons, which later proved useful for explaining the emission and absorption spectra of atoms (Units 10 and 11).

In this Section, we are going to look again at the processes that involve transfer of energy and momentum in interactions.

2.1 The interactions between macroscopic objects, nuclei and electrons

We need say little about collisions between macroscopic objects, such as motor cars or billiard balls. Such interactions are well understood and unambiguously described in terms of *exchanges* of energy and momentum, subject to the *conservation* laws. Experiments in nuclear physics show that the same is true for the collisions between atoms, nuclei and electrons. For example, in Figure 13 a fast electron enters a bubble chamber and collides with another electron, belonging to an atom of the liquid. The second electron is hit sufficiently hard (that is, it is

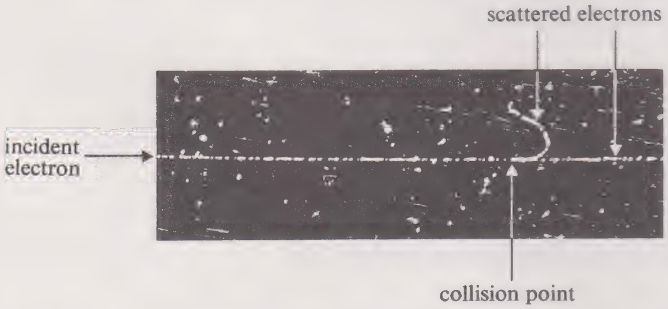


FIGURE 13 A collision of an electron (entering from the left) with an atom of liquid in a bubble chamber. One atom's electron is knocked away from its original atom and forms a track of its own.

given enough energy and momentum) to break loose from its parent atom and to form a trail of bubbles showing its path. By measuring the tracks of the particles involved in the collision, it is found that total momentum and energy are conserved, just as in the collisions of large objects. (You will learn more about bubble-chamber tracks in Unit 31.)

2.2 The interaction of electromagnetic radiation with macroscopic objects, atoms and electrons

In Unit 9 you were presented with one particular kind of interaction between electromagnetic radiation and metals, known as the photoelectric effect. In this interaction, electrons are released from the metal by absorbing energy supplied by the light of suitable frequency.

2.2.1 The photoelectric effect

The experimental observations, collected in detailed investigations of this effect, can be summed up in the following points:

1 For each given metal there is a characteristic *threshold frequency* f_t. Regardless of its intensity, a radiation of frequency less than f_t does not release even a single electron. Radiation of frequency higher than f_t, on the other hand, releases electrons immediately, no matter how weak its intensity.

2 The number of electrons released per second is proportional to the intensity of radiation (Figure 14).

3 The maximum kinetic energy of released electrons increases as the *frequency* of radiation increases (but is independent of intensity).

4 The relationship between the maximum kinetic energy of photoelectrons and the frequency of radiation is the same for all metals (Figure 15—all lines for different metals have the same slope, they differ only in the threshold frequency f_t).

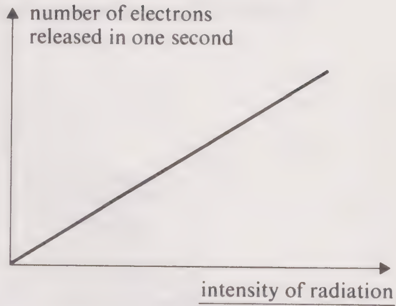


FIGURE 14 The number of photoelectrons released in one second by radiation of frequency higher than threshold frequency increases with increasing intensity of radiation (number of electrons $\alpha I \alpha A^2$).

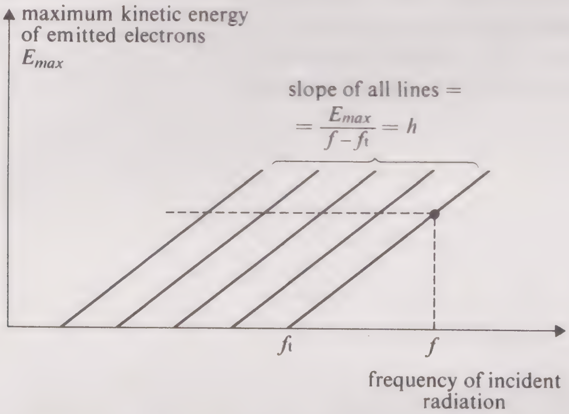


FIGURE 15 The photoelectric effect on different metals.

Points 1 to 4 are a short summary of more detailed explanations given in Unit 9. If you need to refresh your memory, go back to Section 6 of Unit 9 and do the SAQs at the end of that Section.

In order to explain these observations, it was necessary to abandon a wave model of light and to accept that, in the interaction with metal, light behaves as a stream of particles, called *photons*. The frequency f that describes the wave before interaction is connected with the amount of energy E carried by one photon taking part in the interaction, by the formula:

$$E = hf \qquad (18)$$

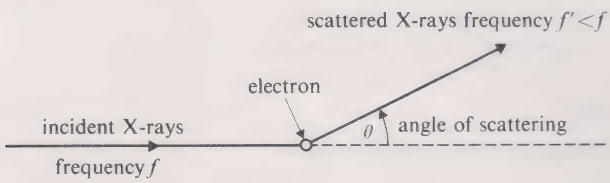
where $h = 6.63 \times 10^{-34}$ J s and is Planck's constant. Our explanation of what actually happens in the photoelectric effect assumes that each photon is *fully absorbed* by an electron. The photon disappears and its energy hf is used in part to free the electron from its bond in the material and the rest is transferred to the electron as kinetic energy.

2.2.2 The Compton effect: do photons have momentum?

The photoelectric effect is an example of a complete transfer of energy from the photons of the radiation to the electrons of the metal. There is, however, another type of interaction between electromagnetic radiation and electrons in solids, in which only *partial transfer of energy* takes place.

This effect is known as the *Compton effect*, after the American physicist A. H. Compton, who observed it first in 1924. He irradiated different materials with beams of X-rays and studied what happened to the frequency and intensity of the initial beam in interactions with the electrons inside materials.

He discovered that X-rays of a single frequency f, travelling towards the target in a very narrow, well-defined beam, were *scattered*, spread out in a very wide range of directions after passing through the target, and that the scattered X-ray radiation had *lower frequencies* than the initial beam. Moreover, there was a clear relationship between the *angle of scattering* θ and the loss of frequency $\Delta f = f - f'$ (Figure 16). The larger the angle of scattering, the lower was the frequency f' of the scattered radiation.



Compton scattering

FIGURE 16 Compton scattering—the scattered X-rays have lower frequency than the incident X-rays.

How can this observation be explained? As you know, X-rays are just another form of electromagnetic radiation, different from light and radio waves only in the frequency of the wave. Yet, here the behaviour of X-rays in the interactions with atomic electrons differs significantly from the photoelectric effect. In the photoelectric effect, the energy of oncoming light is absorbed in units of *full* photons. In the Compton effect, only *part* of the energy of each X-ray photon is transferred to the electron. The amount of energy transferred to the electron is clearly equal to:

$$\Delta E = h\Delta f = h(f - f') \qquad (19)$$

All this seems reasonable enough, but why should the *direction* in which the X-ray photon continues depend on the amount of energy it loses in the interaction? This cannot be explained by energy changes alone, since energy does not depend on the direction of propagation. Photons of radiation of frequency f carry energy hf wherever they go, just as the kinetic energy of a car depends only on its speed, not on the direction of travel.

The only way to make sense of the Compton effect is to accept that *photons must carry some momentum*, and that during the interactions with target electrons both energy and momentum are transferred, in accordance with the conservation laws.

This creates a problem: how would you describe the momentum of a photon? For ordinary particles, momentum was defined as the product of their mass and their velocity. But photons *do not have any mass*, in the meaning defined in Unit 3. You cannot catch a photon and compare its inertial and gravitational properties with those of a test object of unit mass. Furthermore, in a given homogeneous medium, electromagnetic radiation of given frequency always travels at *one constant speed*, you cannot accelerate it or slow it down. Clearly, the momentum of a photon cannot be defined in terms of mass and velocity. Yet, the Compton effect leaves no other option but that photons must have momentum. Thus, experimental necessity forced physicists to accept that *momentum is a more general, more fundamental concept* than those of mass and velocity. Only in the special case—of objects having a measurable mass m and moving with velocities $\vec{v}$, very small compared with the velocity of light—can the momentum be written as $m \cdot \vec{v}$. In the case of electromagnetic radiation, the *momentum of the photons* is deduced from the results of the Compton experiment, using the laws of conservation:

$\lambda = h/p$ $\quad p = \frac{h}{\lambda}$ $\quad c = \lambda f$ $\quad \lambda = c/f$

momentum of a photon $p = hf/c$

$p = \frac{hf}{c}$

$$\boxed{\textit{the magnitude} \text{ of the momentum is } p = \frac{hf}{c}} \qquad (20)$$

the direction of the momentum is that in which the radiation is propagated.

In formula 20, h is Planck's constant (6.63×10^{-34} J s), f is the frequency of radiation, and c is the speed with which the radiation of frequency f travels in the given medium.

With this definition of momentum, the results of the Compton scattering experiment can be described in the same way as a collision between two macroscopic objects. Figure 17 shows this in a diagramatic form. Note in particular that because the oncoming radiation had no component of momentum in the direction y, the two y-components after the interaction must add to zero (remember Section 1.1). Appendix 1 (optional) spells out mathematical equations for the conservation of energy and momentum in Compton scattering.

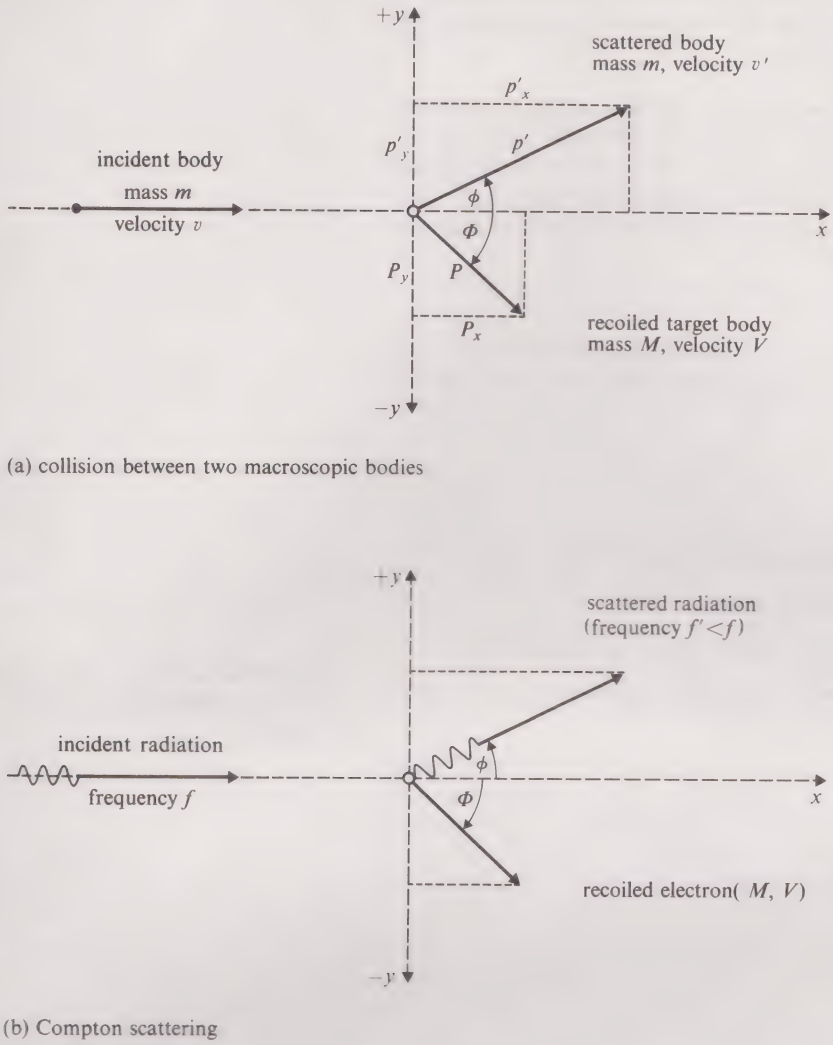


FIGURE 17 Diagram illustrating the analogy between the Compton scattering and the collision of two bodies (energy and momentum are conserved).

In Section 1 we found that the *propagation* of light and the propagation of beams of particles can both be described by the same mathematical formulae in terms of a *wave model*.

Now, in Section 2, we have just reached the conclusion that the *interaction* between electromagnetic radiation and electrons can be described by the same mathematics as collisions between macroscopic objects, in terms of a *particle model* (exchanges of momentum and energy, subject to the laws of conservation).

Thus we have a problem of terminology. Should we refer to electrons, protons, atoms, etc., and to photons, as *waves* or as *particles*? As you would have seen many times by now, this is not a problem of which of the two descriptions is right and which is wrong. Neither of them is completely right or completely wrong; either of them is *appropriate* in certain circumstances and not in others.

In order to circumvent this problem of terminology and to emphasize the underlying uniformity in the behaviour of different forms of physical reality, we introduce a new term: QUANTUM. This term covers electrons, protons, neutrons etc., as well as photons, and it is understood that quanta behave like particles in interactions, but travel like waves. Thus, *quantum theory* is the theory of behaviour of quanta. You have already met some aspects of this theory when you studied the photoelectric effect, the Compton effect and de Broglie waves. In the rest of this

quantum

quantum theory

$c = f\lambda \quad \lambda = \frac{c}{f}$

$p = \frac{hf}{c} \quad \frac{c}{f} = \frac{h}{p} = \lambda$

$\lambda = \frac{h}{p}$

Unit we shall develop some very general principles of quantum theory. In Unit 30 some of these principles will be applied to the understanding of the structure of the atom.

SAQ 6 Compare formula 20, for the momentum of a photon, with the de Broglie formula for the wavelength of a particle (Section 1.2, equation 5). Are they the same or not? What can you conclude from this about the way in which the concept of wavelength was applied to particles (de Broglie) and the concept of momentum to electromagnetic radiation (Compton)? In your comparison, consider only the magnitude of momentum, do not worry about the direction arrows.

SAQ 7 A beam of X-rays of frequency $f = 10^{18}$ Hz undergoes Compton scattering by stationary electrons. Assume, for simplicity, that after one particular scattering both the electron and the scattered photon move in the same direction as the initial X-ray beam. What is the magnitude of the momentum of the electron after the collision, if the frequency of scattered radiation is $f' = 9.9 \times 10^{17}$ Hz?

(Take $h \approx 6.6 \times 10^{-34}$ J s; $c = 3 \times 10^{8}$ m s^{-1}.)

SAQ 8 Will the electron, with the momentum calculated in SAQ 7, be moving very fast, or very slowly? Make a guess first; then do an approximate calculation taking $m_{el} \approx 10^{-30}$ kg.

2.3 Summary so far

To our previous statement (Summary of Section 1) that:

1 *All radiation is propagated as travelling waves,*

we are now in position to add a second, equally general, statement that:

interactions are described by exchanges of energy and momentum between quanta

2 *The interaction of any radiation with matter may be described in terms of exchanges of energy and momentum between quanta.*

Thus, all forms of matter behave in a uniform way in each of the two different types of situations—in propagation and in interactions. The wavelength λ and the frequency f are concepts associated with the wave-like behaviour that characterizes propagation. The momentum $\bar{p}$ and the energy E, on the other hand, are concepts associated with the quantum behaviour of matter, as revealed in interactions.

The de Broglie formula:

$$\lambda_{dB} = \frac{h}{p}$$

and the previously established relationship between energy and frequency:

$$f = \frac{E}{h}$$

provide, through Planck's constant h, a *quantitative link between the two types of behaviour* (propagation and interaction), and between the two models that describe this behaviour (waves, particles).

To be able to describe all propagation and all interactions so succinctly, you must surely agree, is immensely satisfying. One of the primary aims of any scientist is so to order observations, that all phenomena can be described in terms of only a few basic principles. Quantum theory certainly embodies some most powerful unifying principles. It is even more than that; as you will see in the remaining Sections, it has deep philosophical implications too.

An aside: is quantum theory relevant to everyday life?

You may have been puzzled by the approach we have taken so far in trying to discuss, in a very general way, the propagation of energy and momentum by different kinds of radiation, and the exchanges of energy and momentum in interactions that led to the loss of distinction between waves and particles. Why

did we not get straight to the point and asked much simpler questions, such as: 'What *is* light made of—waves or particles?' and 'What *are* electrons—particles or waves?'

Well, the problem is that questions like these *cannot* be answered. You saw in Unit 9 that our conceptual models of light reflect correctly only some *aspects of the behaviour* of light in certain well specified circumstances. All we have done so far in this Unit is to show that even 'particles' are not quite what they appear to be in ordinary circumstances. There are situations in which the actual *behaviour* of objects such as electrons, protons, atoms and molecules, just cannot be understood if we try to vizualize them as tiny solid balls of well-defined shape and size. Indeed, the main reason why the birth of quantum theory was an intellectually painful process—extended over a period of a quarter of a century—was the insistence on asking, and trying to answer, unanswerable questions about the *nature* of these objects. It is now accepted that quantum theory is a *theory of behaviour*, not a theory of the inherent attributes of material objects.

Before you dismiss quantum theory as being evasive, stop and think for a moment about some very simple and clear statement, such as, for example: 'This object is blue'. You may think that this statement tells you something about the nature of the object and not just about its behaviour. But does it?

If you were to examine the same object in a room illuminated by a red lamp, it would no longer be blue, it would appear black. If you heated it to a sufficiently high temperature, it would glow red, or even white. So, the apparently clear statement above does not tell you anything about the nature of the object. In fact, expressed in this form, it is almost useless. The only reason why it seemed so clear is that, without saying so, or being aware of it, you took the *meaning* of that statement to be something like this: 'When irradiated with ordinary daylight (white light), this object reflects only the blue part of the spectrum'. So, it is no more than a statement of behaviour. It is no different in kind from the statements of quantum theory. There is no more evasion or contradiction in saying: 'Electrons behave as waves in some situations and as particles in others', than there is in realizing that one and the same object can be both blue and black.

Continuing the aside, let us consider another general point about quantum theory—its relevance to everyday life. You may feel that we are making much ado about nothing. After all, if streams of marbles cannot produce any detectable diffraction effects, why did we bother to show that they too are described correctly by quantum theory? Smaller, invisible, particles show diffraction effects—but who cares? How does this help us to understand practical problems?

Well, there is a staggering multiplicity of chemical substances in living organisms and in the inorganic materials in the Universe. Yet all this multiplicity results from combinations of a relatively small number of chemical elements. Furthermore, the structure of different elements can be broken down into very few building bricks, from which they are all made according to some common rules. The existence of these building bricks, as well as the rules of their combination, are elucidated by quantum theory.

And it is not just understanding for its own sake. Understanding is a necessary prerequisite for making use of things. It is no exaggeration to claim that some of the most far-reaching advances in modern science and technology—including chemistry and biochemistry—have grown from the applications of quantum theory. Without this theory, there would be no nuclear power stations (and no nuclear bombs, of course), no transistor radios and telecommunication satellites, no understanding of the genetic code, no computers and pocket calculators, etc. For better or worse, quantum theory is very relevant to our everyday life.

3 Probability waves

In the photoelectric effect, the number of electrons released in one second from an illuminated metal plate was found to be proportional to the intensity of the light shining on it. If each electron is ejected by a collision with a photon, the intensity of the light falling on a given surface must be proportional to the number of photons arriving on that surface in one second. Now suppose light passes through

a narrow slit, as in Figure 18. In this diagram, as in all the subsequent diagrams showing arrangements of slits, it is to be understood that the long dimension of the slit is at right angles to the plane of the paper; the gap shown represents the width of the slit.

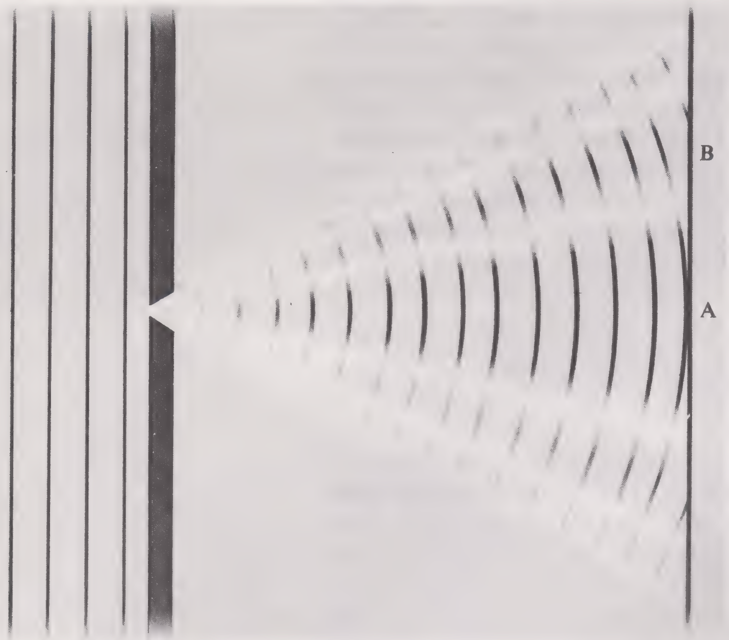


FIGURE 18 Diagram illustrating the diffraction of light at a slit. There are regions of maximum intensity on the screen at positions such as A and B.

Earlier, it was shown that for this experimental arrangement the form of the intensity distribution after the slit is that of Figure 6. From this distribution, it can be seen that, for example, the intensity of the central maximum is about 20 times that of the first diffraction maximum. It therefore follows that 20 times the number of photons arrive on the screen in a small region opposite the slit (position A of Figure 18) as will arrive on an area of equal size situated at the first diffraction peak (position B). Thus, the wave behaviour determines how many photons arrive on each part of the screen (the quantum behaviour of course determines how the photons interact once they arrive).

So far so good. But what if the slit is opened and closed very rapidly so that the average energy transmitted in the time it is open is less than the energy of a single photon? Does only a fraction of a photon arrive at the screen? We appeal once again to experiment. It is found that when light of very low intensity is allowed to pass through an opening for a tiny fraction of a second (in one particular version of the experiment the time was one nanosecond), either a photon with the whole energy hf, arrives at the screen, or none at all.

Given then that only a single photon's worth of light arrives at the screen, what happens to the intensity distribution? Does the photon smear itself out to form a diffraction pattern?

No. The photon does *not* spread itself out—that much is once again established by experiment. When a single photon arrives, it strikes one, and only one, position on the screen. Whereabouts on the screen it will strike cannot be predicted.

> Are there any regions of the screen to which you might guess the photon would *not* go?

All that can be known in advance is that it will arrive at a point on the screen where the diffraction pattern (*calculated* from the wavelength and the size of the slit) would be non-zero, i.e. the photon is liable to arrive anywhere on the screen other than at positions lying on a calculated diffraction minimum. (Note the stress on the word 'calculated'. When dealing with a single photon, there is no *observable* diffraction pattern; it can only be *calculated* from the known wavelength and the size of the slit.)

Likewise, a second photon will go to some position on the calculated diffraction pattern. Subsequent photons do the same, and gradually, with time, the form of a diffraction pattern materializes, as in Figure 19.

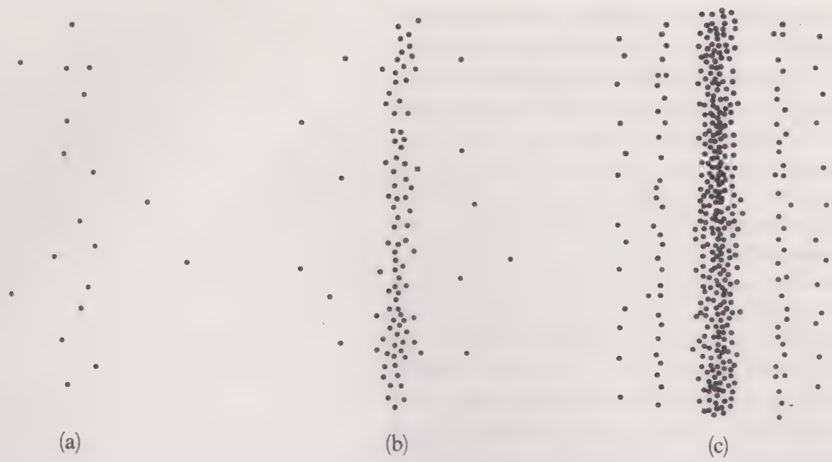


FIGURE 19 As more and more photons pass through the slit and reach the screen, the diffraction pattern gradually builds up and becomes recognizable as such.

Figure 20 shows the same kind of effect with electrons. These photographs were taken with an electron microscope. They show the gradual build up of a diffraction pattern for electrons diffracted at the edges of a small hole.

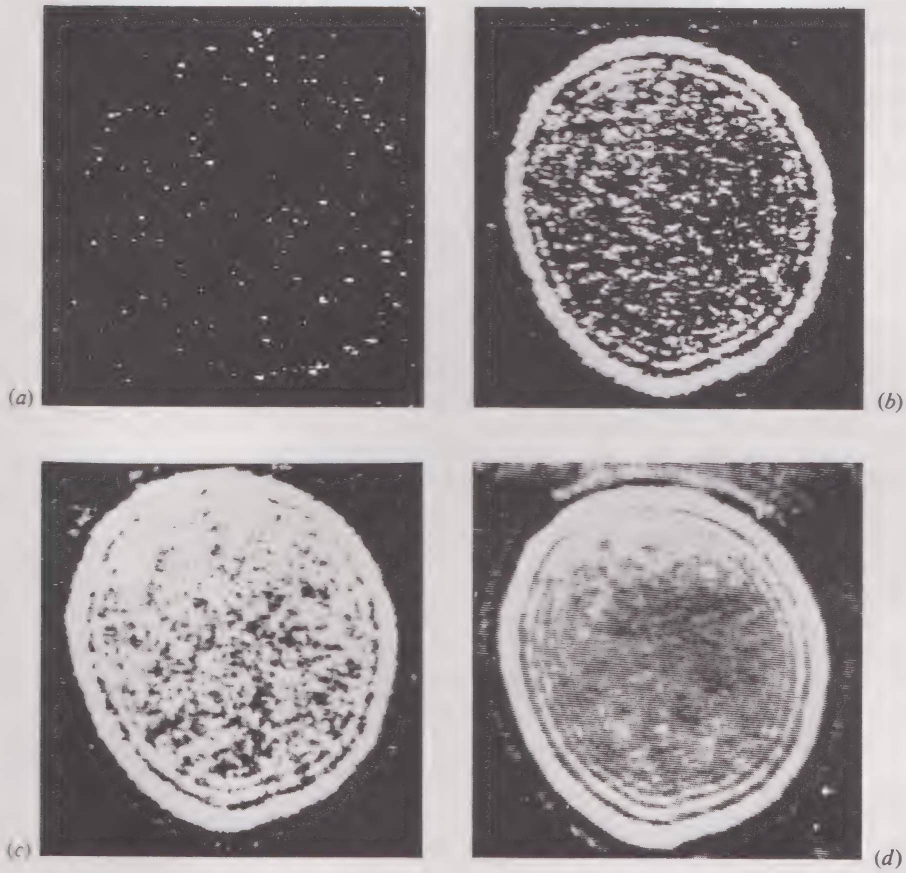


FIGURE 20 Photographs taken with a very low intensity electron beam in an electron microscope. Each dot is an image formed by a single electron. The exposure times were (a) 1/25th second, (b) 10 seconds, (c) 1 minute, and (d) 2 minutes. Note how the form of the diffraction pattern becomes clearer as the number of electrons in the pattern increases.
(Magnification × 125 000.)

Thus, the calculated diffraction pattern describes the *probability* of a photon arriving at various parts of the screen. Where exactly any individual photon will go cannot be predicted—we can only calculate, with the help of wave mathematics, the odds on it going to one place rather than another.

> What are the chances of a given photon going to region A of Figure 18, compared with its chances of going to region B?

The intensity of the diffraction pattern in region A is 20 times greater than in region B. Thus, the number of photons arriving there is 20 times greater, and consequently the chance that any given photon will arrive there is 20 times greater.

To talk of being able to predict only the odds on the outcome of an experiment may well strike you as very strange. After all, you have probably been told in the past that physics is an exact science. If a physicist repeats an experiment that he has often performed before, he is expected to be able to specify in advance what the outcome will be. To a large extent, of course, this remains true; if he takes a quantity of gas and halves the volume, keeping the temperature constant, he knows this will double the pressure; if he doubles the temperature difference across the faces of a sheet of material, he knows this will double the heat transmitted through it, etc. In these, as in many other examples that you can no doubt think of for yourself, the outcome is known—at least to a high degree of precision. However, you should note that these examples deal only in gross features; they each involve the average effects produced by an enormous number of atoms—in one case, the momentum imparted by many gas atoms to the walls of a container averaged over a period of time; in the other, the energy transferred between large numbers of vibrating atoms as they jostle each other. In this respect, all these examples are similar to the experiment in which a substantial amount of light is allowed to pass through the slit. In that case also, the physicist is able to predict the outcome of the experiment, i.e. he knows in advance that, given enough photons, he will get a diffraction pattern of a certain size and intensity distribution. This will allow him to predict, for example, that 20 times as many photons will arrive at region A as at region B (Figure 18).

But, when he is asked to predict the behaviour of a *single* quantum, he is unable to do it. In this kind of situation, arguments that relate only to statistical averages do not help. The physicist finds himself in a rather similar position to that of the 'opinion pollster', whose statistical methods can forecast with reasonable accuracy the percentage vote for each political party averaged over the nation, but do not tell him how a given individual will vote.

We repeat: *there is no way of predicting where exactly a particular quantum will arrive.*

That is a statement we do not expect you to accept lying down! So let us try and anticipate your objections. Consider the following argument:

> The nature of the experiment you have described has not been specified sufficiently well; all you have told me is that the light passes somewhere through the slit. Now, if I were allowed to make a really thorough study of the initial photon, I could determine very precisely its original trajectory and could then work out which atom (or atoms) in the rim of the slit it would interact with. I could then (in theory at any rate) calculate precisely how the photon would scatter from this atom, and hence determine its final direction. That would tell me exactly where it would go on the screen.

This, incidentally, is an example of a *thought experiment*; in these imaginary experiments, *the question of whether the experiment is practicable or not is ignored. All equipment and measurements are assumed to be as close to the ideal as can be imagined.* You have met thought experiments before, in Unit 4, for example, where you worked out how the velocity of a seismic wave should depend on the density and rigidity of the medium through which it travels. Thought experiments have always been regarded as particularly useful in clarifying the meaning and significance of quantum theory.

> At this point we throw down a challenge.
>
> Think back over the material that has so far been presented in this Unit, and see if you can formulate two or three convincing arguments to show what is wrong with the assumptions or line of reasoning in the thought experiment suggested above.

If you think that this is difficult, most of the Course Team would probably agree with you! But don't give up just yet. Try the three clues below and make a genuine attempt to develop the line of reasoning suggested in each before reading on.

> *Clue 1*
> The precise study of the collision between the photon and the atom would presumably involve a detailed knowledge of the struck atom. Have you been told anywhere that the diffraction of light through a slit in a barrier depends on the nature of the material of the barrier?

Clue 2
Is it possible, even in principle, to determine precisely the trajectory of the initial photon? Try designing an experimental arrangement for doing this.

Clue 3
Suppose you opened up a second slit in the barrier very close to the first—what effect would this have on the calculated intensity distribution? Can you reconcile this change of probabilities with a photon whose final direction can be precisely specified by the way it scatters on the rim of only one of the holes?

Clue 1 refers to the possibility of predicting accurately the path of each individual photon, if it were known which particular atom it struck. But, if this were the case, the outcome of such a collision would surely depend on the kind of atom involved in the collision—how heavy it is, what shape it has, how it is oriented and how strongly it is bound to its neighbours. This would mean that the distribution of light after the passage through narrow slits of the same size would depend on the *material* from which they are made. This just is not true, the diffraction pattern depends *only* on the wavelength and on the width of the slit.

Clue 2 explores the possibility of determining precisely the trajectory of the initial photon, before it collides with an atom. One way of attempting this would be to place a tiny hole H in front of the light source, with the aim of narrowing the beam of light before it enters the slit S (Figure 21). In one extreme position, you could

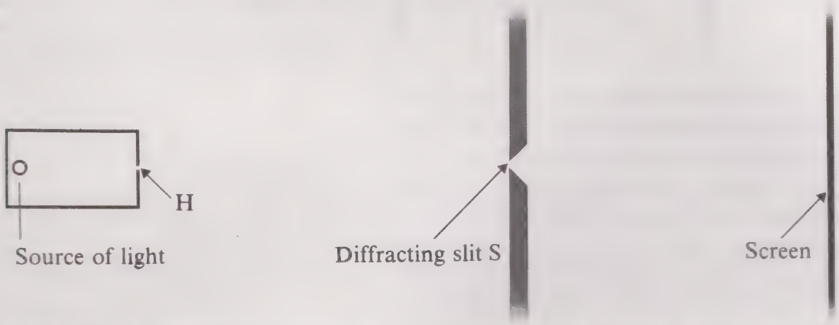


FIGURE 21 A hole H introduced in front of the source in an attempt to produce a well-defined beam of light.

bring the slit right against the hole (Figure 22). If the hole H were to be exceedingly small, you would then know *where* the photon was as it entered the slit.

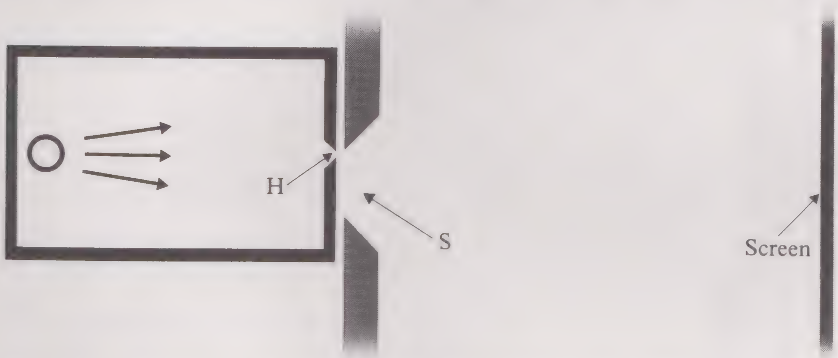


FIGURE 22 The hole H is made very small and is brought up close to the slit S. In this manner, the position of the photon at S is well defined—but not its direction.

But an exceedingly small hole produces an exceedingly wide diffraction pattern! You would know the position of the photon as it entered S, but there would be no way of telling in which *direction* it would be moving. So it would be quite impossible to use the law of conservation of momentum to work out the outcome of the collision of this photon with any particular atom in S.

FIGURE 23 The hole H is removed a long way from S. In this arrangement, the direction of the photon at S is well defined—but not its position.

In order to fix the direction of the photon, you might try to move the slit S very far away from the source and the hole H (Figure 23). Now, because the slit subtends such a small angle with respect to the source, the direction of the photon as it passes between the edges of S is defined quite satisfactorily. Unfortunately, just

because its direction is defined so narrowly, the photon must be passing through the opening of the slit alongside its edges, parallel to thousands and millions of atoms. Which *one* of them will it strike? By specifying the direction of the photon you have sacrificed the knowledge of the position where the interaction took place.

It is obvious that the diffraction of light makes it quite impossible, even in principle, to specify a precisely determined trajectory of a photon (that is, its position *and* momentum at any instant). Any argument that *assumes* such knowledge, must therefore be fallacious.

Clue 3 offers the final and perhaps the most convincing refutation of the possibility to predict accurately the path of individual photons in the diffraction experiments. Suppose a second slit S′, of the same width, is opened very close to S and parallel to it (Figure 24). This is the double-slit arrangement, familiar from Unit 9.

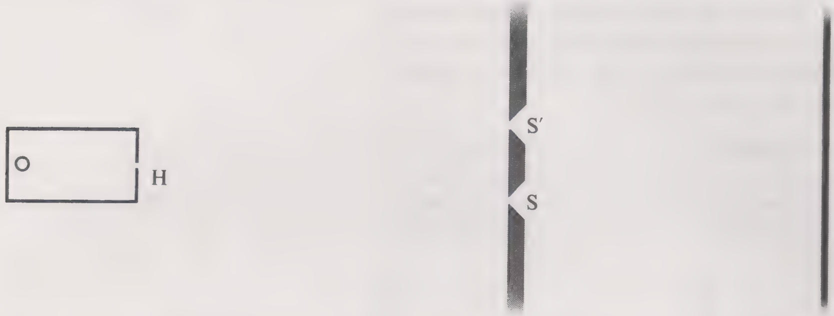


FIGURE 24 The opening of a second slit S′ produces a radical change in the intensity distribution at the screen on the right.

The intensity distribution on the screen is now *radically different* from what it was when only one slit was open, as you can see by comparing Figure 25 with Figure 6. This new pattern is not a simple combination of two single-slit patterns (only slightly displaced relative to each other). There are now regions of zero intensity (the minima in Figure 25) where the screen was illuminated before. This means that a photon may arrive at one of such regions when only one slit is open, but is unable to do so when a second slit is added. How can the addition of the second slit possibly influence the interaction of the photon passing through the first one?

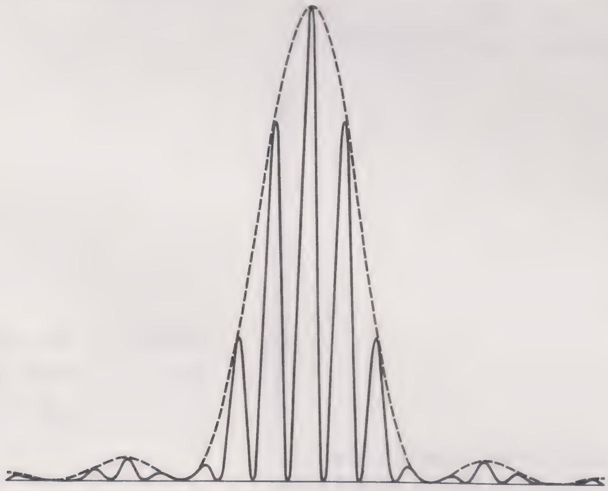


FIGURE 25 The intensity distribution on the screen resulting from light passing through a double slit.

Clearly, any idea of a photon behaving as an ordinary particle with well-defined position and momentum must be abandoned. Consequently, it is impossible, even in principle, to make accurate predictions about *individual* photons. The only way to understand the experiments is in terms of *probability*. The diffraction pattern produced by a large number of photons is accurately predictable, but the path of each individual photon is not. This lack of determinism in the behaviour of material systems at the sub-microscopic level was the greatest intellectual hurdle that physicists had to overcome. You are in good company if you feel apprehensive and uneasy about it!

Although most of the argument in this Section referred to the diffraction of light, you know already that the same principles apply to any kind of radiation, to the

beams of any quanta. Any diffraction pattern can be interpreted in terms of a wavelength, provided the width of a single slit, or the spacing between the slits, is known. Thus it makes sense to associate with the *motion of any beam of quanta* some kind of a wave, which determines the size and shape of the diffraction pattern and therefore the probability with which one single quantum may arrive at any particular spot within the pattern.

probability waves

Such waves are known as *travelling probability waves*. Their wavelength is λ_{dB} and is defined by the momentum of the quanta (cf. Section 1.2). The *wavelength* of the travelling probability wave determines the angular *separation* of the diffraction maxima within the pattern. The relative *brightness* of individual maxima and minima is in turn related to the *amplitude* of the travelling probability wave at each particular point in the pattern.

Travelling probability waves behave in the same way mathematically, as travelling water waves or travelling sound waves. That is, two waves can combine to give constructive interference at certain positions and destructive interference at others (Figure 11). But what about the nature of these probability waves? What is it that is 'waving' in them? Here the analogy breaks down (you have been warned before not to take analogies too far!). Water waves and sound waves are disturbances in real media, but a travelling probability wave is only a mathematical concept, a model, that does not have any medium. The study of ordinary waves tells us something about the state of the medium in which they are propagated. Probability waves, on the other hand, give us no such information. Instead, they tell us of *the state of our knowledge about the propagation of quanta.*

Perhaps you find an abstract concept such as this unpalatable. But why should you? You probably did not flinch when, in earlier Units, we discussed the Earth's orbit around the Sun—but what is an orbit? If you went out into space you would see no line drawn on anything. An orbit is an abstract concept useful for describing our knowledge of successive positions of the Earth. If you are happy to use a mathematical line in space as a convenient summary of our knowledge of the Earth's motion, why not mathematical waves?

4 Heisenberg's uncertainty relations

The situation described in the last Section is quite frustrating. When the position of the photon was fixed (as in Figure 22), the direction of the photon was not. When the experimental arrangement was altered to remedy this (as in Figure 23), it was found that the newly-acquired knowledge of the photon's direction had been bought at the expense of losing knowledge of the photon's position.

The trouble stems from the wave behaviour governing propagation and its probabilistic interpretation. Moreover, because wave behaviour is the underlying principle governing *all* propagation, it is only to be expected that this kind of frustration will be encountered again and again; it is not some peculiarity associated only with photons, it is associated with all quanta.

In this Section, we shall study this fundamental problem of observation in greater depth. We shall investigate to what extent it is possible simultaneously to measure two properties of a quantum: its position and its momentum. For this purpose, we shall again use a slit. The act of measurement is considered to take place at the instant the quantum arrives at the slit. Although we are sticking to our example of diffraction at a slit, you should realize that what is being discussed has universal significance; we could just as well have chosen some other experimental arrangement with which to illustrate the point we wish to make.

4.1 The position–momentum uncertainty

In Figure 26, a drawing of the experimental arrangement is combined with one of the intensity distribution, to remind you of the appearance of the diffraction pattern on the screen when the slit is illuminated with monochromatic radiation from a distant source. The *y*-axis has been drawn at right angles to the length of the slit. (Remember that the length of the slit is taken to be perpendicular to the plane of the paper.)

At the instant at which the quantum arrives at the slit, its position along the y-axis is uncertain; it is only known that it must lie somewhere between the extreme edges of the slit. If the width of the slit is d, then the *uncertainty in the position* of the quantum along the y-axis—call it Δy—at the instant it arrives at the slit is given by:

$$\Delta y = d \tag{21}$$

At this instant, the quantum is 'diffracted', i.e. receives a sideways push resulting in it subsequently arriving at the far screen at a position L, which may be other than position O directly opposite the slit from the source (Figure 26). How large the push will be, no one can say. The photon must, therefore, be regarded as having an uncertain momentum in the y-direction as well as an uncertain position. (Let us emphasize that we are here talking of uncertainty of both position and momentum at the moment the quantum arrives at the *slit*. Do not confuse this with the uncertainty of position on the *screen*; the spread of positions of quanta on the screen will only be used as a convenient measure of the uncertainty of the y-component of momentum at the slit.)

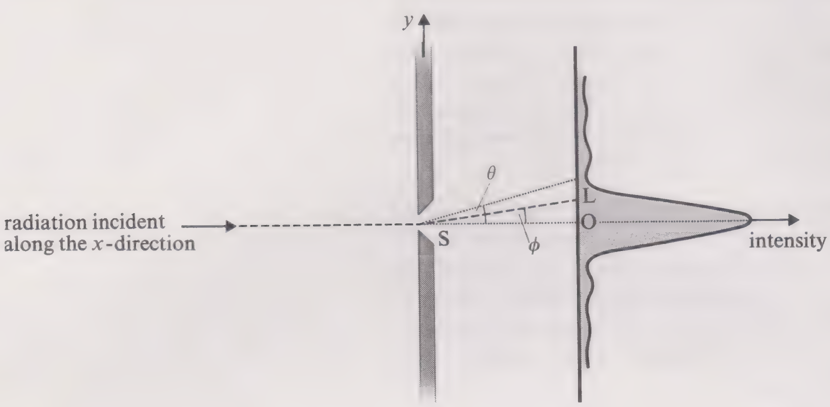


FIGURE 26 A drawing combining the experimental arrangement and the intensity distribution on the screen appropriate to radiation of a single wavelength.

If the point L is such that the direction SL makes an angle ϕ to the original direction of the quantum, then the magnitude of the y-component of momentum $\vec{p}$ would be $p \sin \phi$, as shown in Figure 27 (remember Section 1.1 and Figure 1). But, as we have said, at the time of the measurement there is no way of telling what exactly the value of ϕ will be. All we can do is to take a look at the observed (or calculated) diffraction pattern and decide on some 'typical' value for the angle.

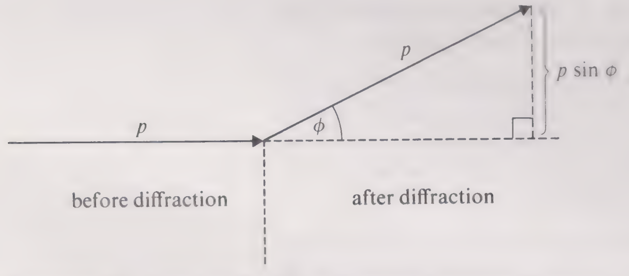


FIGURE 27 After diffraction, the quantum acquires a sideways momentum of magnitude $p \sin \phi$.

In Unit 9 you did an experiment with the diffraction of light by a single slit. You saw that the distribution of intensity after the slit had the form shown in Figure 26 (or previously in Figure 6). Most of the diffracted light went into the broad central lobe, which was spread symmetrically about the straight-through direction. If you repeated the experiment with slits of different width d and with light of different wavelength λ, you would find that the *angular width* of this lobe 2θ (Figure 26) depended on d and λ in the following way:

$$\sin \theta = \frac{\lambda}{d} \tag{22}$$

SAQ 9 What is the angular width of the main diffraction lobe for a slit of width $d = 10^{-5}$ m and light of $\lambda = 436$ nm?

If most of the quanta are diffracted by the slit within a spread of $\pm\theta$ about the x-direction (Figure 26), it seems reasonable to regard:

$$\Delta p_y = p \sin\theta \tag{23}$$

as a 'typical' value for the *uncertainty in the magnitude of the sideways momentum* p_y that the quantum acquired within the slit. It is obvious that this uncertainty is closely related to the uncertainty in the position, that is, to the width d of the slit. Substituting into equation 23 from equations 22 and 21 gives:

$$\Delta p_y = p\frac{\lambda}{d} = p\frac{\lambda}{\Delta y}$$

and therefore,

$$\Delta y \Delta p_y = p\lambda \tag{24}$$

The two quantities on the right-hand side of this equation are not independent. For any quantum, they are related by the de Broglie formula 5. Equation 24 can, therefore, be rewritten thus:

$$\Delta y \Delta p_y = p\left(\frac{h}{p}\right) = h \tag{25}$$

This last equation was derived by choosing, to some extent arbitrarily, the 'typical' values for Δy and Δp_y. One could argue that not *all* quanta would have an uncertainty of Δp_y as given by equation 23. Recognizing this element of arbitrary choice in specifying Δy and Δp_y, we shall finally write equation 25 in a less rigid form:

$$\boxed{\Delta y \Delta p_y \sim h} \tag{26}$$

uncertainty relations for position and momentum

where $\sim$ means, in this context, 'of the order of'. This is *Heisenberg's uncertainty relation*—or at least one form of it. Note that in this form it refers only to the uncertainty in the position and momentum in one particular direction. Similar relations hold true for the other two axes of coordinates (Section 1.1, Figure 1—components of momenta)

$$\boxed{\begin{array}{c}\Delta x \Delta p_x \sim h \\ \Delta z \Delta p_z \sim h\end{array}} \tag{26$'$}$$

In essence, these uncertainty relations do not tell you anything more than is already contained in the previous Sections. You have already seen the impossibility of a simultaneous specification of both position and momentum (for example, in the discussions of the travelling wave packet and of the probability waves). All that the Heisenberg's uncertainty relations do is to give an approximate, *quantitative estimate* of the product of these uncertainties: it is *of the order of Planck's constant* $h = 6.63 \times 10^{-34}$ J s.

The most important thing to bear in mind is that the impossibility of an accurate simultaneous measurement of the two quantities has *nothing to do with the accuracy of measuring instruments*, or with faults in the design of experiments. It is an impossibility 'in principle', one which reflects the behaviour of nature, not a deficiency of technology.

When you look at relations 26 and 26$'$ again, you will realize that, within the constraint of the overall uncertainty (which is the order of h), it is possible to trade-off the accuracy of one of the two quantities against the other. This gives the physicist some considerable element of choice in the design of experiments. It is possible to measure the position of quanta more accurately, when this is important, provided that the corresponding increase in the uncertainty of momentum can be tolerated (and vice versa, of course).

SAQ 10 A beam of quanta approaching along direction x (as in Figure 26) is diffracted by a slit of width $d = 10^{-6}$ m. What is the typical uncertainty in the magnitude of the y-component of the momentum of the quanta as they emerge from the slit? (Take $h \approx 10^{-33}$ J s.)

SAQ 11 Suppose that the diffracted quantum in SAQ 10 was an electron of mass 10^{-30} kg. What would be the uncertainty in the magnitude of its velocity along y-direction, Δv_y? (Assume that $p = mv$ can be used.)

It is really somewhat unfortunate that the relations 26 and 26′ carry the name 'uncertainty' relations. The student of quantum theory may well feel that if the professional physicists feel 'uncertain' about what they are doing, there is little hope for him! For this reason the term 'indeterminacy' is sometimes preferred instead of 'uncertainty', to emphasize that these relations are something more than just semi-quantative expressions of frustration. They are a positive tool for describing some basic features of Nature.

Perhaps you still feel it goes against common sense to believe that one is unable simultaneously to measure both the position and the momentum of an object to any desired precision, given that the measuring apparatus is ideal. Until some such time as the material of this Unit finds its way into the primary school curriculum, it will continue to be regarded as against common sense, and students will spend hours trying to find a way round the uncertainty relation. It is something we all apparently have to get out of our systems at one time or another. In Radio 15, you will hear Heisenberg's own description of Einstein's refusal to accept the uncertainty principle and his (unsuccessful) attempts to devise experiments to refute it.

To give you an idea of the pitfalls lying in wait for those who would outwit Heisenberg, we analyse a few approaches in Appendix 2.

> **Study comment** Like the other two Appendices at the back of this Unit, this is optional material and will not be assessed. However, students interested in higher level physics courses are advised to study these Appendices at some stage, either now or in spare time after completing this Course.

We shall here merely draw your attention to two points that emerge from the discussion in that Appendix:

First, the observer, in making an observation, always disturbs the system he is observing. He cannot play a passive role, i.e. obtain his information without actively disturbing the system. This in itself is nothing new; classical ideas of measurement also recognize that an observer interferes with the object under investigation. What *is* new in quantum theory is that the disturbance is unpredictable—the measurement cannot be suitably corrected to take account of the disturbance.

Second, the observer has to make a choice between different courses of action—*either* to concentrate on precision of momentum, *or* precision of position, *or* some kind of compromise. Having decided on a particular kind of measurement, the unpredictable disturbance associated with the act of taking that measurement denies the observer the opportunity of ever gaining the information he *could* have obtained had he taken one of the alternative courses of action. For example, he could consider the alternatives whereby he either uses a small slit to measure the position of a quantum precisely, or a large slit (so keeping the diffraction effects small) to measure the momentum precisely. If he decides to use the small slit, then the very act of measuring the position of the quantum produces a disturbance (a large diffraction) that destroys the information on the quantum's momentum that he *could* have obtained had a large slit been used instead.

Although in this Section we have concentrated on the uncertainty relation for momentum and position, it would be a mistake to think that only these two quantities are affected by it. There are other pairs of variables linked in a similar way and, for any such pair, the choice of experiment predetermines the accuracy with which either of the two can be measured. One important example is the uncertainty relation governing the measurements of time intervals and energy.

4.2 The time–energy uncertainty relation

In the experiment where a slit was used to measure the position and momentum component of a quantum, *when* exactly was the measurement made? You may immediately answer: at the instant the quantum arrived at the slit. But how would one estimate the instant of arrival?

Perhaps the most direct way is to fix a shutter onto the slit as in Figure 28a. The slit could then be opened for only a limited time (Figure 28b). In order for the y-position and momentum of the quantum to be measured, the quantum has to arrive within the time interval for which the slit is open. This interval can be made as small as one likes. In this way, it is possible to fix exactly the instant at which the measurement is made.

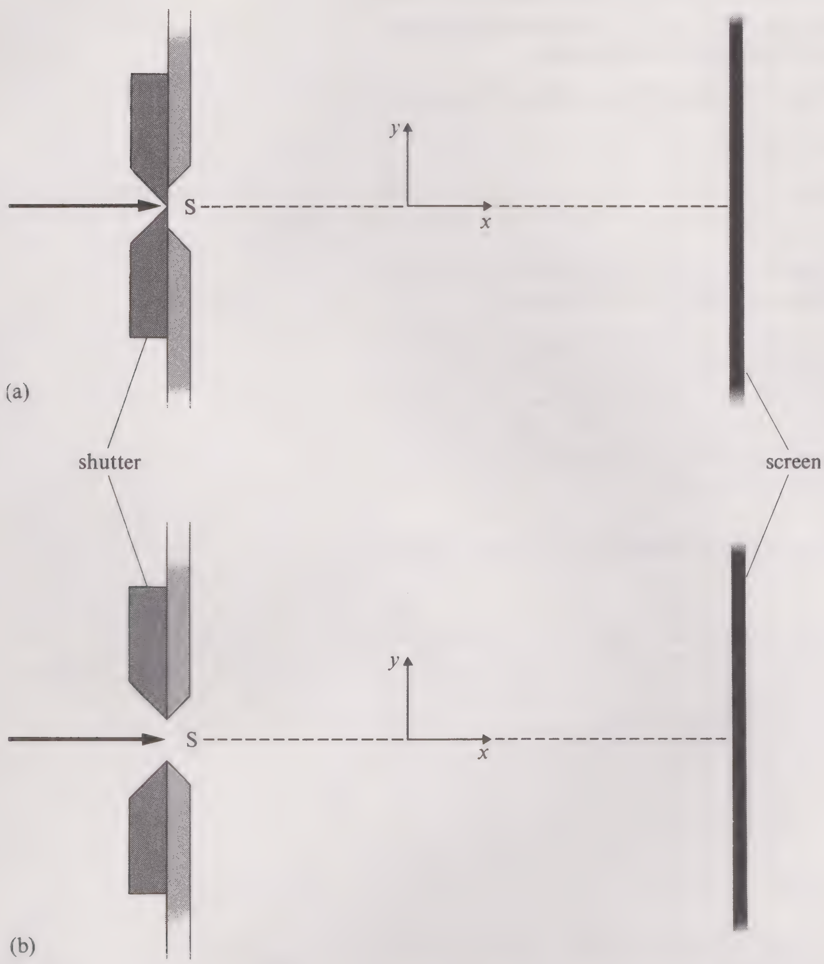


FIGURE 28 (a) A shutter is now placed over the slit through which the radiation is to pass. (b) The shutter can be opened to admit the radiation.

However, the adoption of this procedure has serious repercussions. If the slit is only open for a short time, the radiation transmitted is no longer in the form of an infinitely long, travelling monochromatic wave train—it is in the form of a travelling wave packet (Section 1.6). But this is quite contrary to our earlier assumption; the diffraction pattern shown in Figure 26 is derived on the basis of a *single* wavelength. If we are dealing with a wave packet, this is the same thing as saying our radiation has many different wavelengths; we are now talking of a *different* experiment.

We can look at it another way. If the instant at which the quantum arrives at the slit is known, it follows that, at that instant, it is known exactly where the quantum is along the incident direction, i.e. along the x-axis (it is at the position S). The wave packet describing its position in the x-direction is therefore a sharp spike with a spread Δx that is almost zero. But, according to the uncertainty relation for this direction (relation 26), a small value of Δx implies a large uncertainty in the momentum in the x-direction, i.e. in the incident momentum p. In the original version of the experiment (as in Figure 26), it was assumed that p was known precisely. We now see that such precision could only be achieved with a permanently open slit (resulting in Δx becoming infinitely large and Δp zero).

To summarize: in the previous Section 4.1 we were concerned only with making a measurement of the position and momentum of the quantum in the y-direction. For this purpose, quanta of precisely known incident momentum p were diffracted by a permanently open slit. If, however, we wish to say something about the *time* at which the measurement is made, it is necessary to perform a *different* experiment—one involving, for example, the opening and closing of the slit.

Although this modification of the original experimental arrangement permits us to say something about the time at which the measurement is made, it affects the incident momentum p and our knowledge of it.

If our knowledge of p is uncertain, this is the same as saying our knowledge of the energy E is uncertain; corresponding to Δp, there will be a ΔE. The value of ΔE (like Δp) will depend upon the time the slit is open. Let us call the latter Δt; it is the time available for the measurement. There exists a relation connecting ΔE and Δt. This is a perfectly general relation and applies to all experimental arrangements (not just the slit arrangements we have been considering).

The uncertainty in the energy of a system ΔE and the time available for measurement Δt are related by:

$$\boxed{\Delta E \Delta t \sim h} \tag{27}$$

uncertainty relations for time intervals and energy

The derivation of this relation for the experimental situations we have been discussing is not difficult, and is carried out in Appendix 3 (optional).

You should note that this form of uncertainty relation, although very similar in appearance to the previous ones (relations 26 and 26′), has a different kind of interpretation. The previous relations were all concerned with measurements of two variables, momentum and position, *at a given instant*; this third one can be understood as a relation connecting the uncertainty in the energy to the length of time available for making that energy measurement.

> How much time has to be available in order to make an *exact* measurement of energy?

An infinitely long time ($\Delta t \to \infty$) is required if the energy is to be measured precisely ($\Delta E \to 0$). How does this come about? Well, in the case of a free electron, say, an exact value of E implies an exact value of p, and this in turn implies an exact value of the wavelength (through de Broglie's relation $\lambda_{dB} = h/p$). Such an exact wavelength is characteristic only of an *infinitely long wave train*; strictly

FIGURE 29 A wave train of finite extension.

speaking the wave form in Figure 29 does *not* have an exactly definable wavelength, even though the distances between successive peaks and troughs between the cut-off points X and X′ are equal.

> Why?

This is because many different continuous wave trains must be superimposed in order to produce destructive interference beyond X and X′ and so give the resultant travelling train a finite extension. For a wave form to have only one characteristic wavelength, it is not sufficient that the distance between successive crests and troughs should be equal; the wave train must also extend to infinity. If one has to check that a passing wave train is infinitely long, one needs eternity to complete this measurement!

The energy–time uncertainty relation is extremely important in connection with atomic physics. In Units 10 and 11, you learnt that electrons can occupy different energy levels within atoms, and that they stay in the higher energy levels for only a limited time before spontaneously returning to a lower energy level with the emission of a photon. The energy of the photon is equal to the energy difference between the initial and final states of the electron and is equal to hf, where f is the frequency associated with the photon. But if an electron occupies a higher energy level for only a limited time, this means the total time available for making a measurement of energy must apply to the state of the electron as it was at some instant within that restricted time interval. Relation 27 says that if one is able, by whatever means, to specify a restricted range of time during which the measurement must have been made (and the fact that what is being studied does not *exist*

outside the time interval is undoubtedly one such means!), then the precision with which the energy can be defined is also limited.

Thus, energy measurements on electrons occupying the same high energy level in different atoms will not agree *precisely*. Certainly, for most purposes, the electronic level can be thought of as having a single characteristic energy—this assumption was implicit in the earlier Units—but strictly speaking this is not so. Lines in the spectrum are not infinitely narrow, they have a finite width $\Delta f = \Delta E/h$!

Finally, in view of what has been said about imprecisely determined energies, we take another look at the law of conservation of energy. On the basis of this law, one expects that repeated measurements of the total energy of a system should yield a constant value. If a long time is available for each of these measurements, each measurement will be very precise and, under these circumstances, it is indeed found that the law of conservation of energy holds. (Similarly, the law of conservation of momentum is also found to hold in quantum interactions where the momenta are determined precisely.) However, if the repeated energy measurements are made in rapid succession, the time available for each is limited and so the precision of each energy measurement is also limited. Under these circumstances, the results of repeated measurements do not give a constant value, but fluctuate with a spread characteristic of the uncertainty ΔE. Such readings do not contradict the law of conservation of energy—they indicate that it holds to within the limited precision permitted by the experiment. There are, however, situations where a physical system can change its state so rapidly that it becomes impossible experimentally to specify its precise energy over short periods of time. (This applies to the unstable atomic nuclei described in Unit 30.)

Because Heisenberg's uncertainty relations apply to different pairs of experimental variables* and because the limit of uncertainty is inherent in the *nature* of experiments and not in their practical and technical limitations, the relations 26 and 27—or their equivalent formulations in words—are sometimes called Heisenberg's uncertainty *principle*.

Heisenberg's uncertainty principle

SAQ 12 Imagine a beam of quanta, moving in direction y towards a slit in a barrier, as shown in Figure 30. The width of the slot in the x-direction is 10^{-8} m; its lengths in the z-direction is infinite.

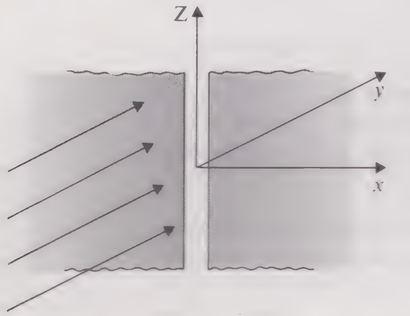


FIGURE 30 A thought experiment with a beam of quanta (for SAQ 12).

1 Is it possible, in principle, to determine the magnitude of the z-component of the momentum of the quanta with absolute accuracy?

2 Is it possible to determine the magnitude of the component of the momentum in x-direction with accuracy better than 10^{-30} Ns?

SAQ 13 Atoms of a certain element are excited by an electric discharge in a fluorescent tube, so that one of their electrons is moved to a particular energy level. The frequency of the emission spectral line, corresponding to the transition from this excited level to the ground state, is carefully studied and it is found to have a width of $\Delta f = 10^{12}$ Hz. What is the corresponding uncertainty in the energy of the excited level? How long, on average, do the atoms live in this particular excited state? (Assume that for the ground state $\Delta E = 0$; $h = 6.63 \times 10^{-34}$ J s.)

*One important pair we have not mentioned is the angular position and the angular momentum of a body rotating about a fixed centre. To discuss this case properly would require more advanced mathematics.

5 Philosophical interpretations of quantum theory

Apart from opening new ways of understanding physical phenomena at the atomic and subatomic levels, the development of quantum theory also provoked a deep philosophical controversy. This concentrated on the problems of causality in world events and on the limits of man's ability to understand, describe and communicate his observations of them.

To put the problem into perspective, let us quote a sentence from the French mathematician and astronomer, Laplace (1749–1827):

> Give me the initial data on all particles and I will predict the future of the Universe.

This sentence sums up very clearly the confidence which nineteenth-century scientists had in the strength and omnipotence of the classical mechanics developed in the previous two centuries. But, perhaps more importantly, it also describes the *intuitive belief* of most people that world events are interconnected in clear links of causes and effects, and that by knowing everything about a particular system at one instant of time one can unambiguously predict its behaviour in the future.

Quantum theory has shattered this belief. As you have seen, it is *impossible* to determine simultaneously both the position and the momentum of a particle with unlimited accuracy. Thus, it is impossible ever to obtain that initial set of complete and accurate data about particles on which Laplace wanted to base his prediction of the future.

There has never been any argument about the *physical content* of quantum theory. The relationships between the parameters of particle (E, $\vec{p}$) and wave (f, λ) models of matter are completely vindicated by all experiments. So are the uncertainty relations between the pairs of parameters such as momentum and position, or energy and lifetime of atomic states. There is also general agreement that quantum theory does not imply that there can be *no* form of predictability of events. As you have seen in the description of the build-up of diffraction patterns (Section 3), it is perfectly possible to *predict* the shape and size of the pattern from the wavelength of radiation and the dimensions of the diffraction grating. What cannot be predicted is the behaviour of *individual* quanta. Thus, quantum theory recognizes causal, predictable connections between an initial and a final observation, providing these arise from the *statistical properties* of large quantum systems.

The philosophical problem arises from the question whether this restriction on the accurate knowledge of individual events is one of principle, inherent in the fundamental structure of the world, or whether it is one of the temporary inadequacy of our theoretical models.

There is one school of thought which believes that quantum theory is a *complete theory*, in the sense that its basic hypotheses about the particle and wave parameters and about the uncertainty relations are ultimate, final reflections of the real world, not capable of any further modification. This view, known as the *Copenhagen interpretation* (several of its major exponents such as Heisenberg, Bohr and Born worked at Copenhagen in 1920s, when quantum theory was being developed) is supported by the following arguments:

—physics does not tell us anything about the world as such, only about our observations of it;

—since any observation of an object is only possible by interfering with it in some way or other in order to produce detectable effects, it is in fact meaningless even to ask questions such as what the object is like when it is not observed (i.e. not interfered with);

—thus, in describing experiments in which electron or X-ray diffraction is observed, it is meaningless to ask what happens between the initial observation—that quantum radiation is being emitted and aimed at a diffraction grating or crystal lattice—and the final observation of the pattern on the screen or in the spectrometer;

—since it is impossible to observe what is happening inside the gratings, there is no hope whatever that this situation can ever be described in terms of concepts and models that were developed entirely on the basis of observed phenomena;

—it is, therefore, a question of one's individual attitude whether one wishes to speculate about such unobservable situations or not; it is not and never will be a scientific question.

Against this viewpoint, a minority of physicists feel very uneasy about such a state of affairs. They believe that the present achievements of quantum theory, as well as its limitations, may well be a temporary, intermediate step in the development of the physical theory of the world. This view is sometimes referred to as the *deterministic*, or 'hidden parameters', *interpretation.* Amongst supporters of this view have been Einstein and de Broglie and, more recently, Bohm*. Their argument is based on the philosophical possibility that one day in the future it might be possible to develop a completely new theory that would subsume the present quantum theory into a new framework of basic concepts and laws and would once again enable physicists to describe individual events and their cause–effect relationships. This implies the existence of some hitherto unobserved parameters of world events—hence the alternative name of this interpretation.

It is not our intention to impose on you one of these two alternatives. The first has the attraction of sticking to what is observable and not bringing in any speculative elements. Against that, the second may well appeal to many who feel the urge for a rigid deterministic order at the level of individual events. And it also seems to keep the door open, although nobody can prove or disprove whether that door leads anywhere or not. 'You pays your money and you takes your choice . . .'

6 Summary of Unit 29

1 Electrons, nuclei, atoms and molecules exhibit observable diffraction effects.

2 The diffraction patterns show that the wavelength associated with these forms of radiation is given by the de Broglie formula $\lambda_{\text{dB}} = h/p$, where p is the magnitude of the momentum and h is Planck's constant.

3 The lack of observable diffraction effects for large objects does not mean that de Broglie's formula cannot be applied to these objects. The wavelength expected on the basis of the formula would be too small for diffraction to be noticed.

4 The mathematics governing light propagation and that governing the motion of bodies have been shown to be equivalent. Thus, not only light (as was seen in Unit 9) but all forms of energy are propagated in accordance with wave theory.

5 If the position of a body is localized in space, the mathematical wave associated with it is also localized in space—it is a wave packet. Such a wave packet has a wavelength that cannot be precisely specified—it is some kind of average of the component waves that are used to build up the wave packet.

6 The photoelectric effect shows that, in interactions between electromagnetic radiation and matter, energy is absorbed in discrete packets called photons. Each photon has energy hf, where f is the frequency of the radiation. You were reminded that discrete energy transfers were also characteristic of the emission as well as the absorption of electromagnetic radiation (Unit 10).

7 Compton scattering shows that the momentum associated with a photon has a magnitude of hf/c, where c is the speed of radiation.

8 All interactions can be described in terms of discrete transfers of energy and momentum in accordance with the principles of conservation of energy and momentum.

9 A study of the behaviour of a single quantum shows that the travelling waves governing the propagation are to be regarded as mathematical probability waves—the intensity of the wave at a given place determines the probability that the quantum will arrive at that place.

10 Because one can only predict probabilities for the various possible results of a measurement, it becomes impossible, even in an idealized experiment, to specify precisely the simultaneous values of the momentum and the position of an object. Along each spatial direction, there exists an uncertainty relation: for example, $\Delta p_x \Delta x \sim h$, where Δp_x and Δx are the uncertainties in the magnitude of the x component momentum and position respectively. These relations were proposed by Heisenberg.

* Do not worry remembering all the names involved, it is the views we want you to think about.

11 A further uncertainty relation connects the uncertainty in the energy of a system ΔE with the time Δt available for its measurement, namely: $\Delta E \Delta t \sim h$.

12 Heisenberg's uncertainty principle draws attention to the fact that when an observer is confronted with a range of possible alternatives for measuring, say, the position and momentum of an object, he can only obtain from each of these alternatives one particular set of information about the two variables. His choice of experimental conditions predetermines this information, in accordance with the uncertainty relations. Each act of observation involves a disturbance of the system.

13 Quantum theory deals only with *observations* of the world—not with a world divorced from the process of observation. According to the Copenhagen interpretation this is no temporary restriction but a fundamental limitation for all time—one cannot ever hope to go further and say something meaningful about what happens *in between* observations. In this sense, it is claimed that quantum theory is complete. Not all physicists accept this claim.

The conceptual diagram, shown in Figure 31, can help you to get an overall view of the material in the Unit and assist you in revision.

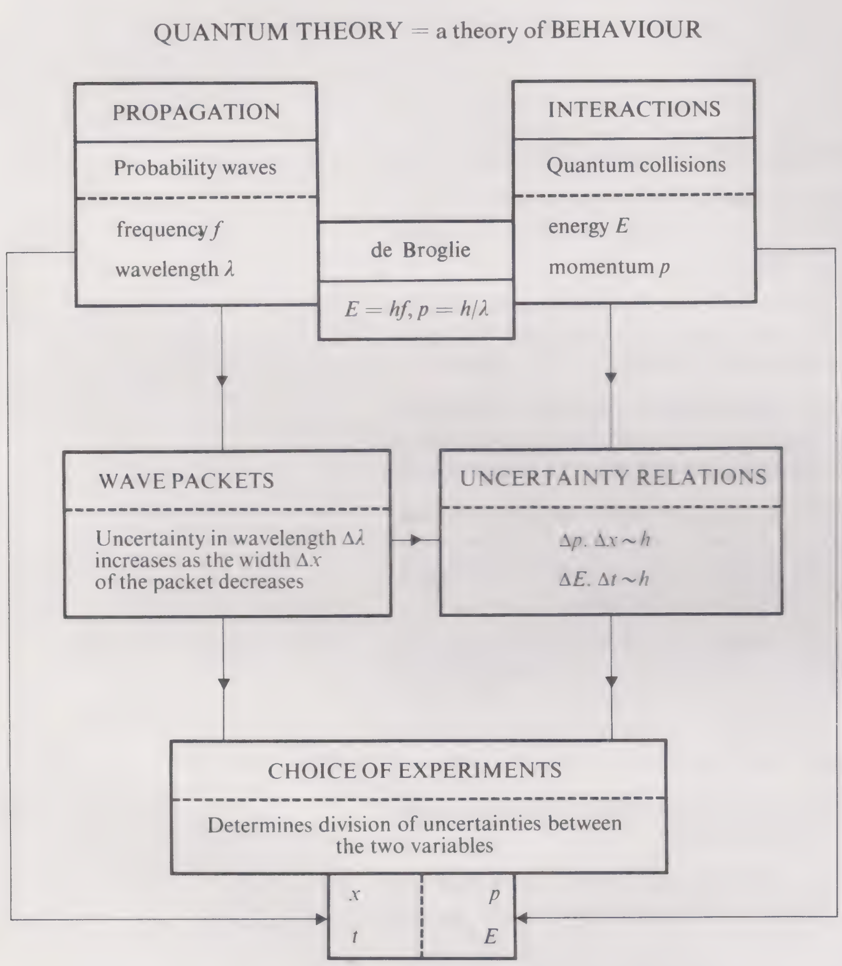


FIGURE 31 Conceptual diagram for Unit 29.

Appendix 1 Energy and momentum conservation in Compton scattering

Figure 17 illustrates the analogy between (b) Compton scattering, and (a) a collision between two macroscopic objects.

Looking at the collision between two objects first, you will note that the incident body was moving along direction x, hence its momentum was:

$$\vec{p} = +p_x \tag{28}$$

After the collision, the scattered body had a momentum $\vec{p}'$ and the recoiled target body a momentum $\vec{P}$. Both momenta have different directions, but both lie in the x–y plane. According to Section 1.1, the law of conservation of momentum means that the components into x- and y-axes must be conserved separately:

$$p'_y + P_y = 0 \tag{29}$$

$$p'_x + P_x = p_x = p \text{ (magnitude of } \vec{p}) \tag{30}$$

Using the angles ϕ and Φ, at which the two objects move with respect to x after the collision, the conservation laws for the two components of momenta can be expressed thus

$$p' \sin \phi + P \sin \Phi = 0 \tag{31}$$

$$p' \cos \phi + P \cos \Phi = p \tag{32}$$

where p', P are the magnitudes of the momenta (the length of the arrows) of the scattered body and the target body respectively and p is the magnitude of the momentum of the incident body.

For macroscopic objects, momenta can be defined by their masses and velocities. Taking into account the conservation of kinetic energy as well, we can finally write down three equations that fully describe the collision:

$$\left.\begin{aligned} \tfrac{1}{2}MV^2 + \tfrac{1}{2}mv'^2 &= \tfrac{1}{2}mv^2 \\ MV \sin \Phi + mv' \sin \phi &= 0 \\ MV \cos \Phi + mv' \cos \phi &= mv \end{aligned}\right\} \tag{33}$$

Looking now at the diagram for Compton scattering, we note that whilst the scattered radiation suffered a loss of frequency, as well as a change of direction, the electron involved in the interaction acquired a momentum $\vec{P} = M\vec{V}$ and energy $\frac{1}{2}MV^2$.

To balance total energy is easy, because you already know that radiation of frequency f behaves as quanta of energy hf. Therefore:

$$hf = \tfrac{1}{2}MV^2 + hf' \tag{34}$$

But where did the momentum of the recoiling electron come from? There is no other possible answer than that it must have been 'carried' by the photons of incident X-rays. So, the conservation of momentum requires that, in analogy with equations 31 and 32, two equations must be satisfied:

$$p'(X) \sin \phi + P \sin \Phi = 0 \tag{35}$$

$$p'(X) \cos \phi + P \cos \Phi = p(X) \tag{36}$$

where $p(X)$ and $p'(X)$ stand for the magnitude of the momentum carried by the X-ray photon before and after Compton scattering respectively, and P is the magnitude of the momentum of the recoiling electron.

It is found that the only way to satisfy these two equations is to assume that the momentum carried by an X-ray photon is related to the frequency of X-ray radiation by the formula:

$$p(X) = \frac{hf}{c} \tag{37}$$

So we can write a set of three equations that fully describe Compton scattering of X-rays on electrons (assuming scattering in one plane):

$$\left.\begin{aligned} \tfrac{1}{2}MV^2 + hf' &= hf \\ MV \sin \Phi + \frac{hf'}{c} \sin \phi &= 0 \\ MV \cos \Phi + \frac{hf'}{c} \cos \phi &= \frac{hf}{c} \end{aligned}\right\} \qquad (38)$$

Appendix 2 Is it possible to get round the uncertainty relations?

First there is the approach that could be described as follows:

> I shall use a slit to study the position of an electron. The slit is made as small as I like, so the electron's position is known. Then, instead of just looking at the general shape of the diffraction pattern and guessing a 'typical' value for the magnitude of the y component of momentum (i.e. $p \sin \theta$, where θ is shown in Figure 26), I wait for one particular electron to strike the distant screen at a particular point L. By measuring the position of L, I can determine the angle ϕ. Then $p \sin \phi$ is the *exact* value of the magnitude of the y component of momentum of this *particular* electron. There is no uncertainty in position and now no uncertainty in momentum.

Hazard a guess as to what might be wrong with this. (The fallacy lies in a basic misunderstanding as to how the uncertainty relation ought to be used.)

This argument involves two separate measurements. The first is a measurement of the position *at the slit*, S, the second, a later measurement *at the screen* used for inferring what the momentum at the slit must have been. The fallacy is that the second measurement refers *back in time* to some former value of the momentum. The uncertainty relation stems from the concept of probability waves and these are concerned only with *prediction*. It is immaterial whether one cares to argue from what is seen on the screen to what happened earlier at the slit; the thesis being developed here is not concerned with that, but with the problem of predicting from measurements made at some instant what a future measurement will reveal. The uncertainty relation is concerned with prediction only and does *not* make statements about the past. In the experiment described above, the uncertainty relation says that the precise knowledge of position gained at the slit excludes the possibility of a precisely determined y-component of momentum at the slit, and therefore excludes the possibility of *predicting* whereabouts on the screen the electron will go. Always be on your guard against misusing the uncertainty relation by applying it to measurements made in retrospect.

The next approach involves a genuine attempt to make a precise prediction:

> Once again, I have a narrow slit so as to determine the position of the electron precisely. But this time I shall use something else to determine the electron's momentum. After it emerges from the hole, and before it strikes the screen, I shall take a look

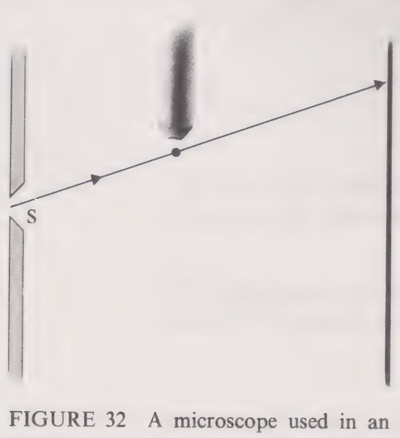


FIGURE 32 A microscope used in an attempt to locate the position of an electron after it emerges from the slit.

> at it in a microscope (as in Figure 32). Having located this second position, I know the direction in which it is travelling and so can predict where it will strike the screen.

What is wrong with this approach?

Clue If you use a microscope, you will need to illuminate the electron to be able to see it. What will you use as illumination? Will the illumination have any effect on the electron?

The use of a microscope demands that the object be illuminated. This requires that light (i.e. at least one photon) should be scattered off it and into the microscope. But in a photon–electron collision there is an exchange of energy and momentum. This means that you are disturbing the electron; it will now go to some other place on the distant screen, not to the one you were trying to determine.

The following is an example drawn from a wide class of approaches all founded on the same basic type of flaw.

> I make the slit in Figure 26 as small as I like so as to fix the position. The barrier with the slit in it is free to move in the y-direction. If it is initially at rest, it will be set moving with velocity v in the y-direction as a result of the momentum imparted to it by the electron, as the electron acquires sideways momentum at the slit. If the mass of the barrier is m, then the momentum is mv. Because momentum is conserved in the y-direction, as in any other direction, the magnitude of the momentum of the barrier mv, must be equal and opposite to the magnitude of the y-component of momentum of the electron, $p \sin \phi$. Therefore $mv = -p \sin \phi$, hence ϕ is determined. I now know precisely the angle ϕ and the position of the electron as it emerged from the slit. Admittedly, I determined the momentum of the electron by a second measurement in retrospect, but in making this estimate I did not disturb the momentum of the electron itself so I can now go on to *predict* whereabouts on the screen the electron will go.

This is indeed a clever argument. The electron is not disturbed after it passes through the slit, and it is a genuine attempt at prediction. If you find it difficult to see what is wrong with it, you may take comfort from the fact that many a seasoned physicist fails to see the weakness too!

Clue The barrier with the slit in it is said to be at rest initially. This means its initial momentum is precisely known to be zero. What does that say about our knowledge of the position of the barrier?

The estimate of the sideways momentum imparted to the barrier is based on the difference between the initial and final momenta of the barrier in the y-direction. These must both be known precisely if the final result is to be precise. But how does one *know* the precise initial value of the momentum of the barrier or indeed the position of the slit? It is implicitly *assumed* in the wording of the argument that this knowledge has been gained through performing a preliminary experiment. Such a preliminary experiment might consist of shining a light on the barrier on two separate occasions. If, for example, the screen is seen not to move between the two observations, then it can be concluded that the initial momentum of the barrier is zero. But how can one be sure that the second time the barrier was illuminated it did not acquire some momentum from the photons? If it did, then the initial momentum of the barrier when the electron arrives at the slit would not be zero. This, in its turn, means that the *preliminary experiment is incapable of determining precisely the initial position of the barrier—and hence the position of the slit in the y-direction.* A precise knowledge of the initial momentum of the barrier, combined with only an imprecise knowledge of the position of the slit, brings us no closer to making a precise prediction.

From this example, you learn that, before considering the experiment proper (in this case the electron arriving at the slit), you have to be quite sure you can set up the apparatus as described. In this example, you could not have done so. A preliminary measurement designed to ascertain the precise momentum of the barrier would inevitably have left the slit with an uncertain position. You have to be very careful to spell out exactly what the function of each part of your apparatus is. If, as in our earlier examples, the slit is used to specify the position of the electron, then it has to be firmly understood that the slit's position is known only as a result of it being *fixed*, and consequently incapable of recoiling and giving information about the electron's momentum. On the other hand, a momentum-measuring barrier designed to recoil, cannot give a precise fix on the position of the electron. *You cannot have it both ways*—you must make your choice and stick to it.

We shall not pursue these arguments any further. You are, of course, free to continue the search for ways round the uncertainty relation if you so wish—but you are advised not to spend too much time on it!

Appendix 3 The derivation of the time–energy uncertainty relation

The magnitude of the momentum p of an object can be expressed in terms of its kinetic energy E_k in the following way:

$$p = mv$$

$$p^2 = (mv)^2 = 2m(\tfrac{1}{2}mv^2)$$

$$p^2 = 2mE_k \tag{39}$$

The uncertainty in kinetic energy ΔE_k has now to be expressed in terms of the uncertainty in momentum Δp. Suppose p is increased by the value of its uncertainty to $(p + \Delta p)$, E_k will be correspondingly increased to $(E_k + \Delta E_k)$ and there will be a relation analogeous to equation 39 connecting these new values of momentum and energy:

$$(p + \Delta p)^2 = 2m(E_k + \Delta E_k) \tag{40}$$

Replacing $2mE_k$ in equation 40 by p^2 (equation 39):

$$p^2 + 2p\Delta p + (\Delta p)^2 = p^2 + 2m\Delta E_k$$

$$\Delta E_k = \frac{1}{2m}[2p\Delta p + (\Delta p)^2] \tag{41}$$

If Δp is small, the second term on the right-hand side is small compared with the first, and can be ignored. Therefore:

$$\Delta E_k = \frac{p\Delta p}{m} \tag{42}$$

If the shutter in Figure 28b remains open for a time Δt, then the measurement can be considered to take place at some moment within that interval. Thus Δt is the uncertainty in the time of the measurement. If the object moves with velocity v in the direction x, then the distance it travels in time Δt is given by:

$$\Delta x = v\Delta t \tag{43}$$

So, if it is known that the object arrives at the slit *sometime* within the interval Δt, one can say that at any *given* instant within the range Δt, the uncertainty in the position of the object is Δx.

We now write down an expression for the product $\Delta E_k\, \Delta t$, using equations 42 and 43:

$$\Delta E_k\, \Delta t = \frac{p\, \Delta p}{m}\frac{\Delta x}{v} = \frac{p}{mv}\, \Delta p\Delta x$$

Thus, because $p = mv$:

$$\Delta E_k\, \Delta t = \Delta p\Delta x \tag{44}$$

But $\Delta p\Delta x \sim h$, therefore:

$$\Delta E_k\, \Delta t \sim h \tag{45}$$

The total energy of an object is the sum of its kinetic energy E_k and its potential energy (i.e. energy that depends on position) U:

$$E = E_k + U \tag{46}$$

If the object is 'free', so that its potential energy is zero (or at least independent of x), the uncertainty in the kinetic energy ΔE_k gives rise to an equal uncertainty in the total energy ΔE. Therefore, substituting ΔE for ΔE_k in equation 45, one obtains the energy—time uncertainty relation referred to in the text as equation 27 (Section 4.2):

$$\Delta E\Delta t \sim h \tag{27*}$$

Objectives

When you have completed the work on this Unit, you should be able to:

1. Apply the laws of conservation of energy and of momentum to simple situations involving head-on elastic collisions between two objects. (ITQs 1–4; SAQs 1 and 2)

2 State the de Broglie formula and perform simple calculations involving the relationship between the momentum of particles and their de Broglie wavelength. (SAQs 3 and 4)

3 State the Maupertuis's principle governing propagation of particles. With reference to the de Broglie wavelength, associated with particles, explain how the principles of Maupertuis and Fermat can be unified into a principle of least number of wavelengths. (Section 1.5; ITQs 5 and 6)

4 Describe what is meant by a continuous wavetrain and a travelling wave packet, and explain why a travelling wave packet does not have a precise wavelength. (Section 1.6)

5 Describe Compton scattering and explain why its understanding requires that photons should have momentum. Perform simple calculations involving the relationship between the frequency of radiation and the momentum of photons. (SAQs 6–8)

6 Explain the meaning of the uncertainty relations between the position and momentum ($\Delta x \Delta p_x \sim h$, and similar for y, z) and between time and energy ($\Delta E \Delta t \sim h$). Perform simple calculations of uncertainties involved in given experimental situations. (SAQs 9–13)

7 Describe in your own words the conceptual development of quantum theory with reference to the diagram in the Summary. (Section 6, Figure 31)

ITQ answers and comments

ITQ 1 This is a question to be answered on the basis of your intuition and general experience. The correct choices are D and J. Perhaps the most important message of this question can be summarized thus: when a light, moving object hits a heavy, stationary target, it always bounces back. The target body moves forward but, because of its larger mass, it will move more slowly than the 'projectile' that hit it.

ITQ 2 Although the actors in this collision are the same, the result will be very different. An object of large mass M will not bounce back when it hits a stationary target object of small mass m. Both objects will move forward, m faster than M. The speed of the large object M will be less than it was before the collision.

ITQ 3 The law of conservation of momentum demands that the total momentum of the two gliders must be the same before and after the collision. Before the collision the total momentum is zero:

$$m\vec{v}_1 + m\vec{v}_2 = +mv - mv = 0$$

Now you must check the total momentum for each of the alternatives A to F.

A Total momentum is zero because both velocities are zero—allowed.
B Total momentum is greater than zero, because *both* velocities have the same positive sign (same direction)—not allowed.
C This is allowed, because $m\vec{v}_1 + m\vec{v}_2 = -mv + mv = 0$.
D, E and F All these alternatives are also allowed by the law of conservation of momentum. Whatever the actual velocities of the two gliders are, the total momentum will always be zero, provided both gliders move with the *same* speed in *opposite* directions ($\vec{v}_1 = -\vec{v}_2$).

This multiplicity of allowed options is clearly at variance with your practical experience. If you repeat an experiment in which two bodies collide, you would always expect—and get—the same result, if the initial conditions were the same (masses, velocities).

ITQ 4 Since the two gliders move without friction and since their collisions are assumed to be elastic (no loss of energy), the only form of energy you need to consider is the *kinetic energy*. Before the collision, the total kinetic energy of the two gliders was:

$$E_k = \tfrac{1}{2}mv^2 + \tfrac{1}{2}m(-v)^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}mv^2 = mv^2$$

Notice that the kinetic energy of a moving body is *independent* of the direction of its motion! Although the *velocities* of the two gliders are opposite (minus sign), the *square* of a negative number is positive.

A check on the total kinetic energy E_k in all alternatives reveals that:

A is not allowed, because $E_k = 0$
B is allowed by conservation of energy, because:

$$E_k = \tfrac{1}{2}mv^2 + \tfrac{1}{2}mv^2 = mv^2, \text{ as before.}$$

C is allowed, because:

$$E_k = \tfrac{1}{2}m(-v)^2 + \tfrac{1}{2}mv^2 = mv^2, \text{ as before.}$$

D is not allowed, because:

$$E_k = \frac{1}{2}m\left(-\frac{v}{2}\right)^2 + \frac{1}{2}m\left(\frac{v}{2}\right)^2 = \frac{1}{2}m\frac{v^2}{4} + \frac{1}{2}m\frac{v^2}{4} = m\frac{v^2}{4}$$

which is four times less than before collision.

E and F are not allowed, because for *any* values of speeds other than v, the total kinetic energy E_k is different from its value before the collision.

Thus, the law of conservation of energy would allow two options, B and C. However, in the answer to ITQ 3 the alternative B was discounted, because it contradicts the law of conservation of momentum. Hence, the one and only alternative that agrees with both conservation laws is C. This is indeed what *always* happens in such a collision between two, *identical* gliders moving in opposite directions—they swap their velocities. (This is true even if their speeds are not the same. If, for example, $\vec{v}_1 = +v$ and $\vec{v}_2 = -2v$ before the collision, then after the collision the velocities will be $\vec{v}_1 = -2v$ and $\vec{v}_2 = +v$. Subscripts 1 and 2 refer to the two gliders. You can easily convince yourself, if you wish, that this extended statement still agrees with the conservation of energy.)

ITQ 5 The formula relating the frequency of a light wave to the energy of the photons of that light is $f = E/h$. The similarity with equation 5 is obvious enough—the two parameters are also related through Planck's constant. More importantly, it is also a relationship between one parameter of a wave model (f) and one parameter of a particle model (E) for the same light.

ITQ 6 Each fraction $\Delta s/\lambda$ in equations 15 and 17 expresses the length of the segment Δs in terms of the number of wavelengths. For example, a wave of wavelength $\lambda = 2\,\text{cm}$, will need five whole wavelengths to cover a segment of length $\Delta s = 10\,\text{cm}$. So,

$$\frac{\Delta s}{\lambda} = \frac{\text{length of segment}}{\text{length of one wavelength}}$$

$$= \text{number of whole wavelengths in the segment.}$$

The whole path $A \rightarrow B$ is the sum of all segments. So the requirement that the numerical value of $\sum_A^B \Delta s/\lambda$ must be a *minimum*, means that the path, selected by the generalized principle of propagation, will be that for which a *minimum number of wavelengths is needed* to cover it. This will of course depend on how λ changes from one place to another (depending on changes of the medium or on the action of forces, such as gravity).

SAQ answers and comments

SAQ 1 The law of conservation of momentum says that the toal momentum of the two skaters is the same before and after the push. The initial momentum is zero, because both skaters are stationary. The same must be true after they have moved apart, so it must be:

$$m_A \vec{v}_A + m_B \vec{v}_B = 0 \qquad \text{and hence } m_A \vec{v}_A = -m_B \vec{v}_B$$

In words: the two velocities are in opposite directions (negative sign) and their ratio is *inversely* proportional to the ratio of the two masses (the lighter skater moves faster). Note that the total momentum is zero because each of the two skaters has a momentum of the same *magnitude* as the other, but of opposite sign ($m_A \vec{v}_A = -m_B \vec{v}_B$).

To find the ratio of the speeds of A and B, we use the above equation, ignoring the minus and vector signs and substituting for m_A and m_B:

$$\frac{v_A}{v_B} = \frac{m_B}{m_A} = \frac{50}{80} = 0.625$$

SAQ 2

A Total momentum is not conserved, since:

$$\vec{p} = mv_1 + mv_2 = -\frac{mv}{2} + \frac{mv}{2} = 0$$

Total kinetic energy after the collision is:

$$E_k = \frac{1}{2}m\left(-\frac{v}{2}\right)^2 + \frac{1}{2}m\left(+\frac{v}{2}\right)^2 = \frac{1}{2}m\frac{v^2}{4} + \frac{1}{2}m\frac{v^2}{4} = m\frac{v^2}{4}$$

This is less than the initial kinetic energy of the two bodies. Hence, neither of the two conservation laws is satisfied:

B In this alternative the total momentum after the collision is:

$$\vec{p} = m\vec{v}_1 + m\vec{v}_2 = m\frac{v}{2} + m\frac{v}{2} = +mv$$

Total kinetic energy is:

$$E_k = 2\left[\frac{1}{2}m\left(\frac{v}{2}\right)^2\right] = m\frac{v^2}{4}$$

Hence, although the total momentum is conserved, the total kinetic energy is not.

C Total momentum $\vec{p} = m\vec{v}_1 + m\vec{v}_2 = 0$ —not conserved.

Total kinetic energy $E_k = 2\left[\frac{1}{2}m\frac{v^2}{2}\right] = m\frac{v^2}{2}$ —conserved.

D Total momentum $\vec{p} = +2m\frac{v}{\sqrt{2}}$ —not conserved.

Total kinetic energy $E_k = m\frac{v^2}{2}$, as in C, —conserved,

E Total momentum $\vec{p} = 0 + m\vec{v}_2 = +mv$ —conserved.

Total kinetic energy $E_k = 0 + \frac{1}{2}mv^2$ —conserved.

F $$\vec{p} = m\left(+\frac{v}{2}\right) + m\left(+\frac{\sqrt{3}v}{2}\right)$$

$$= +mv\left(\frac{1+\sqrt{3}}{2}\right) \quad \text{—not conserved.}$$

$$E_k = \frac{1}{2}m\frac{v^2}{4} + m\frac{3v^2}{4} = \frac{1}{2}mv^2\left(\frac{1}{4} + \frac{3}{4}\right) \quad \text{—conserved.}$$

G $$\vec{p} = m\left(-\frac{v}{2}\right) + m\left(+\frac{3v}{2}\right) = +mv\left(\frac{3}{2} - \frac{1}{2}\right) \quad \text{—conserved.}$$

$$E_k = \frac{1}{2}m\frac{v^2}{4} + \frac{1}{2}m\frac{9v^2}{4} = \frac{1}{2}mv^2\left(\frac{1}{4} + \frac{9}{4}\right) \quad \text{—not conserved.}$$

SAQ 3 The de Broglie wavelength λ_{dB}, associated with electrons travelling at $10^5\,\text{m s}^{-1}$, can be calculated by direct substitution into equation 5:

$$\lambda_{dB} = \frac{h}{p} = \frac{h}{mv} \approx \frac{10^{-33}\,\text{J s}}{10^{-30}\,\text{kg} \times 10^5\,\text{m s}^{-1}}$$

So the order of magnitude of λ_{dB} is:

$$\lambda_{dB} \approx 10^{-8}\,\text{m}$$

By comparison, visible light has wavelengths from about 400 nm to about 700 nm, that is, to the nearest order of magnitude:

$$\lambda_{light} \approx 1000\,\text{nm} = 10^{-6}\,\text{m}$$

However, for any kind of wave motion, diffraction effects become observable only if the gaps in a barrier, the spacing between transparent slits in a grating, or the separation between adjacent atoms in a crystal, are *of the same order of magnitude as the wavelength* of the wave motion, *or less.*

Thus a grating with the spacing of 10^{-6} m is fine for light, but too coarse for a beam of electrons ($\lambda_{dB} \approx 10^{-8}$ m).

SAQ 4 Substituting $m = 10^{-2}$ kg, $v = 1\,\mathrm{m\,s^{-1}}$ and $h = 6.63 \times 10^{-34}$ J s into the de Broglie formula 5 gives:

$$\lambda_{\mathrm{dB}} = \frac{h}{mv} = \frac{6.63 \times 10^{-34}\,\mathrm{J\,s}}{10^{-2}\,\mathrm{kg\,m\,s^{-2}}} = 6.63 \times 10^{-32}\,\mathrm{m}$$

Clearly, with the wavelength of this order of magnitude, a gap of width $d = 10\,\mathrm{mm} = 10^{-2}\,\mathrm{m}$ cannot possibly produce any observable diffraction effects! (Remember, diffraction effects become obvious only when $\lambda \approx d$.)

SAQ 5 According to the diffraction formula for a grating the first diffraction maximum occurs at an angle θ_1 given by the equation:

$$\sin\theta_1 = \lambda/d$$

Substituting λ_{dB} from SAQ 4 gives:

$$\sin\theta_1 = \frac{6.63 \times 10^{-32}\,\mathrm{m}}{10^{-2}\,\mathrm{m}}, \text{ hence } \theta_1 = 6.63 \times 10^{-30} \text{ radians.}$$

(For very small angles, $\sin\theta = \theta$ in radians.) So, if you could follow a beam of marbles, diffracted at such a small angle, all the way up to the edge of the observable Universe (about 10^{25} m), its direction would be displaced from the initial straight line by less than one-tenth of a millimetre!

SAQ 6 For X-rays, as for any other forms of electromagnetic radiation, the speed $c = f\lambda$. Substituting this into equation 20 gives:

$$p = \frac{hf}{\lambda f} = \frac{h}{\lambda}$$

which is, indeed, formally the same as the formula for the de Broglie wavelength:

$$\lambda_{\mathrm{dB}} = \frac{h}{p}$$

This agreement proves that the extension of momentum to electromagnetic radiation is consistent with the previous extension of wavelength to particles.

SAQ 7 The difference between the magnitudes of the momentum p of the incident photon and the momentum p' of the scattered one, is clearly:

$$\Delta p = p - p' = \frac{h}{c}(f - f')$$

Substituting numerical values gives:

$$\Delta p = p(\text{electron}) = \frac{6.6 \times 10^{-34}\,\mathrm{J\,s}}{3 \times 10^{8}\,\mathrm{m\,s^{-1}}}(10^{18} - 9.9 \times 10^{17})\,\mathrm{Hz}$$

Hence $p(\text{electron}) = 2.2 \times 10^{-42}\ \mathrm{N\,m\,s/m\,s^{-1}} \times 0.1 \times 10^{17}\,\mathrm{s^{-1}}$ (remembering that 1 joule = 1 N m and that hertz (Hz) is the name for one cycle per second, $\mathrm{s^{-1}}$). So the final result for the momentum of the recoil electron is:

$$p(\text{electron}) = 2.2 \times 10^{-42} \times 10^{16}\,\mathrm{N\,s} = 2.2 \times 10^{-26}\,\mathrm{N\,s}$$

SAQ 8 An electron with momentum 2.2×10^{-26} N s will be moving quite fast, as becomes obvious by the substitution of its mass into the formula for the magnitude of its momentum:

$$p(\text{electron}) = m_{\mathrm{el}} v$$

and hence $$v = \frac{p(\text{electron})}{m_{\mathrm{el}}} \approx \frac{2.2 \times 10^{-26}\,\mathrm{N\,s}}{10^{-30}\,\mathrm{kg}}$$

The electron will be moving with the velocity of about $2.2 \times 10^{4}\,\mathrm{m\,s^{-1}}$. This is fast by comparison with everyday speeds, but still very slow by comparison with light.

SAQ 9 Substituting into equation 22 gives:

$$\sin\theta = \frac{436 \times 10^{-9}\,\mathrm{m}}{10^{-5}\,\mathrm{m}} = 436 \times 10^{-4} = 0.0436$$

This corresponds to $\theta = 2.5° = 0.0436$ radians (note that for such small angles $\sin\theta = \theta$). Thus the width of the lobe is $2\theta = 0.0872$ radians, or 5°.

SAQ 10 The width of the slit d represents the uncertainty in position along direction y, Δy. So, from the uncertainty relation $\Delta y \Delta p_y \sim h$, it is easy to calculate the corresponding uncertainty Δp_y:

$$\Delta p_y \sim \frac{h}{\Delta y} \sim \frac{10^{-33}\,\mathrm{J\,s}}{10^{-6}\,\mathrm{m}} = 10^{-27}\,\mathrm{N\,s}$$

(Remember 1 joule = 1 newton metre.)

SAQ 11 $\Delta p = m\Delta v_y$ and therefore:

$$\Delta v_y = \frac{\Delta p_y}{m} = \frac{10^{-27}\,\mathrm{N\,s}}{10^{-30}\,\mathrm{kg}} = 10^{3}\,\mathrm{m\,s^{-1}}$$

As you can see this uncertainty is not exactly negligible!

SAQ 12 1 Since the slit is not restricted in the z-direction, there can be no certainty at all about the position of quanta in that direction ($\Delta z = \infty$). Consequently, the uncertainty relation would allow the magnitude of the momentum p_z to be measured with any accuracy (Δp_z could be zero).

2 The uncertainty relation $\Delta p_x \Delta x \sim h$ determines the maximum possible accuracy of Δp_x. Substituting $\Delta x = 10^{-8}$ m shows that the minimum possible value for Δp_x is about 6×10^{-26} N s. Hence an accuracy of 10^{-30} N s or better can never be achieved in this situation.

SAQ 13 The width of the emission line obviously corresponds to the spread of energies, carried by individual photons, ΔE. This, in turn, is the same thing as the uncertainty in the energy of the level, so:

$$\Delta E = h\Delta f = 6.63 \times 10^{-34} \times 10^{12} = 6.63 \times 10^{-22}\,\mathrm{J}$$

The average life-time Δt of the excited state is related to the width of the energy level ΔE, so:

$$\Delta t \sim \frac{h}{\Delta E} = \frac{6.63 \times 10^{-34}\,\mathrm{J\,s}}{6.63 \times 10^{-22}\,\mathrm{J}} = 10^{-12}\,\mathrm{s}$$

By the way, the assumption that $\Delta E = 0$ for the ground state was necessary for any deductions to be made from the observed width Δf. If both atomic states had finite uncertainties in energy, both would contribute to the width of the spectral line.

0 335 08066 9
The Open University Press